On an elliptic system with singular nonlinearity

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Abstract

We discuss the existence and regularity of solutions to an elliptic system, whose basic example is:

$$\begin{cases} -\Delta u = \frac{f(x)v^\theta}{u^\theta} & \text{in } \Omega, \\ -\Delta v = g(x)(1+u)^\gamma & \text{in } \Omega, \\ u > 0, v > 0 & \text{in } \Omega, \\ u = 0, v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $0 \leq f(x) \leq g(x) \in L^m(\Omega)$ are not identically zero, and $\theta, \gamma > 0$ are fixed constants.

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1 Introduction

This short note analyzes existence and regularity of solutions to the following system of elliptic equations in bounded domain $\Omega \subseteq \mathbb{R}^N$:

$$\begin{cases} -\text{div}(M(x)Du) = \frac{f(x)v^\theta}{u^\theta} & \text{in } \Omega, \\ -\text{div}(M(x)Dv) = g(x)(1+u)^\gamma & \text{in } \Omega, \\ u > 0, v > 0 & \text{in } \Omega, \\ u = 0, v = 0 & \text{on } \partial\Omega, \end{cases} \tag{S}$$

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where $\theta > 0$ and $f(x), g(x)$ are nonnegative functions not identically zero, such that

$$f(x) \leq g(x) \in L^m(\Omega), \text{ with } m \geq 2,$$

$M(x)$ is a matrix valued function satisfying

$$\alpha|\xi|^2 < M(x)\xi \cdot \xi \text{ and } |M(x)| \leq \beta,$$

for every $\xi \in \mathbb{R}^N$ and $x \in \Omega$ a.e.

The literature on semilinear elliptic equations is extensive, for a summary of results see for example Gilbarg and Trudinger (2001); Boccardo and Croce (2013) and the references therein.

In Boccardo and Orsina (2010), the authors discuss existence and regularity of solutions to the equation:

$$-\Delta u = \frac{f(x)}{u^\gamma} \quad \text{in } \Omega, \tag{2}$$

where $\gamma > 0$ is a fixed constant and $f \in L^m(\Omega)$, for $m \geq 1$. Existence and regularity are proved under different assumptions on m, using mainly the Maximum Principle and truncation methods as methods of proof. They prove, among other things, that if $m > \frac{N}{2}$ then there exists a bounded solution $u \in L^\infty(\Omega)$ to (2). As we shall see below, this result generalizes to the system considered in this note.

In Boccardo and Orsina (2020), existence and regularity is studied for the system

$$\begin{cases} -\text{div}(A(x)Du) + u = -\text{div}(uM(x)Dv) + f(x) & \text{in } \Omega, \\ -\text{div}(M(x)Dv) = u^\theta & \text{in } \Omega, \end{cases} \tag{3}$$

where $f \in L^m(\Omega)$, A, M are uniformly elliptic matrices and $\theta < \frac{2}{N}$ is fixed. The authors prove, using truncation methods, a general existence theorem and various regularity results depending on m. In particular, they proved that if $m > \frac{N}{2}$, then there exists a bounded solution pair $(u, v) \in [L^\infty(\Omega)]^2$ to (3).

In this note, using the Maximum Principle and truncation methods, we analyze existence and regularity of solutions to System S. More precisely, we show that if $m > \frac{N}{2}$, and γ satisfies

$$0 < \gamma < \frac{2}{m}\left(\frac{2m - N}{N - 2}\right), \tag{4}$$

then:

(1) If $\theta \leq 1$, then system (S) has a solution $(u, v) \in [\mathbf{H}_0^1(\Omega) \cap L^\infty(\Omega)]^2$.

(2) If $\theta > 1$, then system (S) has a solution $(u, v) \in (H^1_{\text{loc}}(\Omega) \cap L^\infty(\Omega)) \times (\mathbf{H}_0^1(\Omega) \cap L^\infty(\Omega))$, moreover $u^{\frac{\theta+1}{2}} \in \mathbf{H}_0^1(\Omega)$.

Notation & Assumptions

- $\Omega \subset \mathbb{R}^N$ is a bounded domain and $N \geq 3$.
- The space $\mathbf{H}_0^1(\Omega)$ denotes the usual Sobolev space which is the closure of $\mathcal{C}_0^\infty(\Omega)$, smooth functions with compact support using the Sobolev norm.

- For $q > 1$, q' denotes the Holder conjugate, i.e. $\frac{1}{q} + \frac{1}{q'} = 1$, and q^* denotes the Sobolev conjugate, defined by $q^* = \frac{qN}{N-q} > q$.
- The letter C will always denote a positive constant which may vary from place to place.
- The Lebesgue measure of a set $A \subseteq \mathbb{R}^n$ is denoted by $|A|$.
- The symbol $\rightharpoonup$ denotes weak convergence.

2 Proofs

A **distributional solution** to system (S) is a pair of functions $(u, v) \in [\mathbf{W}_0^{1,1}(\Omega)]^2$ such that both fv^θ and $g(1+u)^\gamma$ are in $L^1(\Omega)$,

$$\forall U \subset\subset \Omega\, \exists C_U \,:\, u \geq C_U > 0 \text{ in } U \tag{5}$$

and

$$\begin{aligned} \int_\Omega M(x) Du D\varphi &= \int_\Omega \frac{f(x) v^\theta}{u^\theta} \varphi \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega) \\ \int_\Omega M(x) Dv D\phi &= \int_\Omega g(x)(1+u)^\gamma \phi \quad \forall \phi \in \mathcal{C}_0^\infty(\Omega). \end{aligned} \tag{6}$$

We start by truncating system (S). Namely, given $n \in \mathbb{N}$, we set $f_n(x) := \max\{f(x), 0\}$ and $g_n(x) := \max\{g(x), 0\}$ and consider the system

$$\begin{cases} -\text{div}(M(x) Du_n) = \frac{f_n(x)(T_n(v_n))^\theta}{(u_n + \frac{1}{n})^\theta} & \text{in } \Omega, \\ -\text{div}(M(x) Dv_n) = g_n(x)(1 + T_n(u_n))^\gamma & \text{in } \Omega, \\ u_n = 0, v_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{T}$$

Proposition 1. *System* (T) *has a solution* $(u_n, v_n) \in [\mathbf{H}_0^1(\Omega) \cap L^\infty(\Omega)]^2$. *Moreover, both* u_n *and* v_n *are positive.*

Proof. The proof is topological, using Schauder Fixed point theorem. Fix $n \in \mathbb{N}$, and define an operator $S : L^2(\Omega) \to L^2(\Omega)$ given by $S(u_n) = w_n$, where $w_n \in \mathbf{H}_0^1(\Omega)$ is the unique solution of

$$-\text{div}(M(x) Dw_n) = \frac{f_n(x)(T_n(v_n))^\theta}{(|u_n| + \frac{1}{n})^\theta} \tag{7}$$

and v_n is the unique solution of

$$-\text{div}(M(x) Dv_n) = g_n(x)(1 + T_n(u_n))^\gamma. \tag{8}$$

Here, uniqueness is guaranteed by Lax-Milgram theorem. Taking w_n as a test function in (7), we obtain:

$$\alpha \int_\Omega |Dw_n|^2 \leq n^{2\theta+1} \int_\Omega |w_n| \tag{9}$$

By Poincare's inequality:

$$\int_\Omega |w_n|^2 \leq C \int_\Omega |w_n|$$

We conclude that

$$\|w_n\|_2 \leq C,$$

Hence, if we denote the ball centered at the origin of radius C by $B_C(0)$, then $B_C(0)$ is invariant under the operator $S(u_n)$.

By the compactness of the embedding $\mathbf{H}_0^1(\Omega)$ in $L^2(\Omega)$, S is also compact. Finally, S as defined is clearly continuous, so Schauder Fixed point theorem confirms that there is $u_n \in \mathbf{H}_0^1(\Omega)$ such that

$$S(u_n) = u_n.$$

Moreover, since the right hand side of both equations are in $L^\infty(\Omega)$, classical elliptic regularity theory guarantees that $(u_n, v_n) \in L^\infty(\Omega) \times L^\infty(\Omega)$. Also, since both u_n and v_n are superharmonic, by the strong maximum principle, they have to be positive in Ω.

We conclude that the pair (u_n, v_n) is a solution to system (T). □

Theorem 2. *If $m > \frac{N}{2}$, and γ satisfies:*

$$0 < \gamma < \frac{2}{m}\left(\frac{2m-N}{N-2}\right), \tag{10}$$

then the sequence v_n is bounded in $\mathbf{H}_0^1(\Omega)$ and also in $L^\infty(\Omega)$. Moreover,

$$u_n \leq v_n,$$

hence u_n is also bounded in $L^\infty(\Omega)$.

Proof. Taking the difference of the two equations of system (T), we have:

$$\int_\Omega M(x)(Du_n - Dv_n)D\varphi = \int_\Omega \left(\frac{f_n v_n^\theta}{(u_n + \frac{1}{n})^\theta} - g_n(1+u_n)^\gamma\right)\varphi \tag{11}$$

for $\varphi \in \mathbf{H}_0^1(\Omega)$. Choosing $\varphi = (u_n - v_n)^+$ and using the ellipticity of $M(x)$ we obtain:

$$\alpha \int_\Omega |D(u_n - v_n)^+|^2 \leq \int_\Omega g_n \left(1 - (1+u_n)^\gamma\right)(u_n - v_n)^+. \tag{12}$$

On the other hand, notice that the right hand side of (12) is always nonpositive, hence:

$$\alpha \int_\Omega |D(u_n - v_n)^+|^2 \le 0 \Rightarrow (u_n - v_n)^+ \equiv 0.$$

We conclude that $u_n \le v_n$ for every $n \in \mathbb{N}$.

Take v_n as a test function in the second equation of (T). Using Young's and Poincare's inequality (with constant $\mathcal{P} > 0$), we obtain:

$$\begin{aligned} \alpha \int_\Omega |Dv_n|^2 &\le \int_\Omega g(v_n + v_n^{\gamma+1}) \\ &= \int_\Omega g v_n + \int_\Omega g v_n^{\gamma+1} \\ &\le C \int_\Omega g^2 + \frac{\alpha}{4\mathcal{P}} \int_\Omega |v_n|^2 + C \int_\Omega g^{(\frac{2}{\gamma+1})'} + \frac{\alpha}{4\mathcal{P}} \int_\Omega |v_n|^2 \end{aligned} \tag{13}$$

$$\therefore \int_\Omega |Dv_n|^2 \le C.$$

Now, the condition $0 < \gamma < \frac{2m-N}{mN}$ together with $v_n \in L^{2^*}(\Omega)$ implies

$$g_n(1 + T_n(u_n))^\gamma \in L^s(\Omega),$$

with $s > \frac{N}{2}$. By Stampacchia's regularity theory, $\|v_n\|_\infty \le C$, and since $u_n \le v_n$, we automatically obtain $u_n \le C$. □

Corollary 3. *If $m > \frac{N}{2}$, and γ satisfies:*

$$0 < \gamma < \frac{2}{m}\left(\frac{2m-N}{N-2}\right), \tag{14}$$

both sequences u_n and v_n satisfy condition (5).

Proof. By the theorem above, $\|u_n\|_\infty \le \|v_n\|_\infty \le C_\infty$ for a constant C_∞ independent of n.

Notice that for every $n \in \mathbb{N}$ and $\varphi \in \mathcal{C}_0^\infty(\Omega)$, v_n satisfies:

$$\int_\Omega M(x) Dv_n \varphi + \int_\Omega v_n \varphi \ge \int_\Omega g_1 \varphi,$$

so by the maximum principle, v_n satisfies condition (5) and also $v_n \ge C_{g_1}$ in Ω, for a constant $C_{g_1} > 0$ independent of n.

Similarly, for every $n \in \mathbb{N}$, u_n satisfies:

$$\int_\Omega M(x) Du_n \varphi + \int_\Omega u_n \varphi \ge \int_\Omega \frac{f_n(x)(T_n(v_n))^\beta}{(C_\infty + 1)^\theta} \varphi \ge \int_\Omega \frac{f_1(x) C_{g_1}^\beta}{(C_\infty + 1)^\theta} \varphi. \tag{15}$$

Therefore, by the maximum principle again, u_n satisfies condition (5). □

The case $\theta \leq 1$

This is simplest case, and existence of solutions can be obtained very easily:
Lemma 4. *Let u_n be the solution obtained in Proposition 1. If $\theta \leq 1$, the sequence u_n is bounded in $\mathbf{H}_0^1(\Omega)$.*
Proof. Take u_n as a test function in the first equation of (T) and using the fact that $u_n \leq v_n \leq C$ a.e. , we obtain:

$$\alpha \int_\Omega |Du_n|^2 \leq \int_\Omega \frac{f_n(x)T_n(v_n)}{(u_n + \frac{1}{n})^\theta} u_n \leq \int_\Omega f v_n u_n^{1-\theta} \leq C\|f\|_1 \tag{16}$$

□

Theorem 5. *If $\theta \leq 1$, $m > \frac{N}{2}$, and γ satisfies:*

$$0 < \gamma < \frac{2}{m}\left(\frac{2m - N}{N-2}\right), \tag{17}$$

then system (S) has a solution $(u,v) \in [\mathbf{H}_0^1(\Omega) \cap L^\infty(\Omega)]^2$.
Proof. Since both u_n and v_n are bounded in $\mathbf{H}_0^1(\Omega)$, we can assume that up to a subsequence $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ in $\mathbf{H}_0^1(\Omega)$. We can easily take the limit in the terms to the left hand side of (S). The only non trivial part is whether or not we can pass the limit on the terms to the right. Notice that for every $\varphi \in \mathcal{C}_0^\infty(\Omega)$:

$$\begin{aligned} &\left|\frac{f_n(x)v_n^\theta \varphi}{(u_n + \frac{1}{n})^\theta}\right| \leq Cf(x) \\ &|g_n(x)(1+u_n)^\gamma \varphi| \leq Cg(x) \end{aligned} \tag{18}$$

By Lebesgue's dominated convergence theorem, we can pass the limit on the right hand side of (S). □

The case $\theta > 1$

If $\theta > 1$, the problem becomes more interesting, since just taking u_n as testing function in the first equation of (T) wouldn't suffice for obtaining estimates. Despite this, we can still obtain estimates in a larger Sobolev space, as the lemma below demonstrates.
Lemma 6. *Let u_n be the solution obtained in Proposition 1. If $\theta > 1$, the sequence u_n is bounded in $H_{loc}^1(\Omega)$, moreover $u_n^{\frac{\theta+1}{2}}$ is bounded in $\mathbf{H}_0^1(\Omega)$.*
Proof. We now take u_n^θ as a testing function in the first equation of (T) to obtain:

$$\alpha\theta \int_\Omega |Du_n|^2 u_n^{\theta-1} \leq \int_\Omega \frac{f_n(x)v_n^\theta}{(u_n + \frac{1}{n})^\theta} u_n^\theta \leq \int_\Omega f v_n^\theta \leq C\|f\|_1 \tag{19}$$

We can simplify the above to

$$\int_\Omega |Du_n^{\frac{\theta+1}{2}}|^2 \leq C\|f\|_1, \tag{20}$$

which confirms that $u_n^{\frac{\theta+1}{2}}$ is bounded in $\mathbf{H}_0^1(\Omega)$. Now, take $u_n\varphi^2$ as testing function, where $\varphi \in \mathcal{C}_0^\infty(\Omega)$ and $K = \text{supp}(\varphi)$, we have:

$$\alpha \int_\Omega |Du_n|^2\varphi^2 + 2\int_\Omega M(x)Du_nD\varphi u_n\varphi \leq \int_\Omega \frac{f_n(x)v_n^\theta}{(u_n+\frac{1}{n})^\theta}u_nv_n^2 \leq \int_\Omega \frac{fv_n^{\theta+2}}{C_k^\theta} \leq C\|f\|_1 \tag{21}$$

Notice that:

$$\alpha \int_\Omega |Du_n|^2\varphi^2 - 2\int_\Omega |M(x)Du_nD\varphi u_n\varphi| \leq \alpha \int_\Omega |Du_n|^2\varphi^2 + 2\int_\Omega M(x)Du_nD\varphi u_n\varphi \tag{22}$$

Simplifying, we obtain:

$$\alpha \int_\Omega |Du_n|^2\varphi^2 \leq C\|f\|_1 + 2\beta\int_\Omega |Du_nD\varphi u_n\varphi| \tag{23}$$

By Young's inequality:

$$2\beta\int_\Omega |Du_nD\varphi u_n\varphi| \leq \frac{\alpha}{2}\int_\Omega |Du_n|^2\varphi^2 + \frac{2\beta^2}{\alpha}\int_\Omega |Dv_n|^2u_n^2 \tag{24}$$

Combining equations (23) and (24), we have:

$$\frac{\alpha}{2}\int_\Omega |Du_n|^2\varphi^2 \leq C\|f\|_1 + \frac{2\beta^2}{\alpha}\int_\Omega |Dv_n|^2u_n^2 \leq C(\|f\|_1 + 1). \tag{25}$$

It follows that u_n is bounded in $H^1_{\text{loc}}(\Omega)$. □

We conclude with the following result, whose proof is identical to the one we gave in theorem 5, hence it will be ommitted.

Theorem 7. *If $\theta > 1$, $m > \frac{N}{2}$, and γ satisfies:*

$$0 < \gamma < \frac{2}{m}\left(\frac{2m-N}{N-2}\right), \tag{26}$$

then system (S) has a solution $(u, v) \in (H^1_{loc}(\Omega) \cap L^\infty(\Omega)) \times (\mathbf{H}_0^1(\Omega) \cap L^\infty(\Omega))$, moreover $u^{\frac{\theta+1}{2}} \in \mathbf{H}_0^1(\Omega)$.

3 Concluding remarks and open questions

The techniques used in this manuscript fail if $N = 2$, because some estimates cease to be true. It would be interesting to see if the arguments presented here can be adapted to include similar results in the plane as well:

Does system (S) have bounded solutions in 2 dimensions?

We could easily increase the sophistication of system (S), if we substitute the second equation by $-\Delta v = \frac{g(x)u^\alpha}{v^\alpha}$, in other words, consider the system:

$$\begin{cases} -\Delta u = \frac{f(x)v^\theta}{u^\theta} & \text{in } \Omega, \\ -\Delta v = \frac{g(x)u^\alpha}{v^\alpha} & \text{in } \Omega, \\ u > 0, v > 0 & \text{in } \Omega, \\ u = 0, v = 0 & \text{on } \partial\Omega, \end{cases} \tag{Q}$$

Does system (Q) have (unique) solutions? Are they bounded?

The main feature of system (Q) is that it is not clear that the condition $u_n \leq v_n$ still holds in this scenario, even if it does, it's not obvious that u_n, v_n would still be bounded, hence proving that they satisfy (5) could be challenging.

Another interesting related problem that could be approached using the techniques presented here is the system:

$$\begin{cases} -\Delta u = \frac{f(x)v^\theta}{u^\theta} & \text{in } \Omega, \\ -\frac{\Delta v}{(1+v)^\gamma} = g(x)u^\alpha & \text{in } \Omega, \\ u > 0, v > 0 & \text{in } \Omega, \\ u = 0, v = 0 & \text{on } \partial\Omega, \end{cases} \tag{W}$$

The degenerate coercivity could potentially interact with the singularity and influence the existence of bounded solutions, hence the following question seems reasonable:

Does system (W) have bounded solutions?

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