On a fully nonlinear k-Hessian system of Lane-Emden type

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Abstract

In this manuscript we prove the existence of solutions to a fully nonlinear system of (degenerate) elliptic equations of Lane-Emden type and discuss a inhomogeneous generalization.

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1 Introduction

We will study the following fully nonlinear (degenerate) Elliptic system:

$$\begin{cases} F_k[-u] = \sigma\, v^{q_1}, & v > 0 \quad \text{in} \quad \mathbb{R}^n, \\ F_k[-v] = \sigma\, u^{q_2}, & u > 0 \quad \text{in} \quad \mathbb{R}^n, \\ \liminf\limits_{|x|\to\infty} u(x) = 0, & \liminf\limits_{|x|\to\infty} v(x) = 0. \end{cases} \tag{S}$$

where $\sigma \in M^+(\mathbb{R}^n)$ is a nonnegative Radon measure, $0 < q_i < k < \frac{n}{2}$ for $i = 1, 2$, and $F_k[u]$ is the k-Hessian operator, defined as the sum of the k-minors of D^2u.

The equation $F_k[u] = f(x,u)$ in a domain $\Omega \subseteq \mathbb{R}^n$ actually is variational. It's the Euler-Lagrange equation of the functional

$$I_k[u] = \frac{1}{k+1}\int_\Omega u_i u_j F_k^{ij}[u] - \int_\Omega F(x,u),$$

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where F_k^{ij} represents the derivative with respect to the entry a_{ij} and $F(x,u)$ an antiderivative with respect to u.

Similarly to the linear case, when we study the variational problem associated with the k-Hessian equation we divide the problem into three cases, the sublinear case, the eigenvalue problem, and the superlinear case, depending on the behavior of $f(x,u)$. In this manuscript we restrict ourselves to the case of systems with $f(x,u) = u^q$, with $q < k$. However, the results presented here could be generalized to other types of $f(x,u)$ having similar growth rate.

In order for this system (S) to make sense in case u is not regular enough, we consider it in the sense of k-subharmonic functions and k-Hessian measures as defined in Trudinger and Wang (1999).

More precisely, an upper-semicontinuous function is said to be *k-subharmonic* in $\Omega \subseteq \mathbb{R}^n$, $1 \le k \le n$, if $F_k[q] \ge 0$ for any quadratic polynomial $q(x)$ such that $u(x) - q(x)$ has a local finite maximum in Ω. In case, $u \in C^2_{loc}(\Omega)$ then u is k-subharmonic if and only if $F_j[u] \ge 0$ for $j \le k$. In particular, all k-subharmonic functions are subharmonic in the usual sense and if $k = n$ then they are also convex.

The set of all k-subharmonic functions in Ω will be denoted by $\Phi^k(\Omega)$.

We recall the following theorem:

Theorem 1 (Trudinger and Wang (1999))**.** *For any k-subharmonic function u, there exists a Radon measure $\mu_k[u]$, called the k-Hessian measure such that*

i. *If $u \in C^2(\Omega)$ then $\mu_k[u] = F_k[u]dx$; and*

ii. *If u_j is a sequence of k-subharmonic functions which converges to u a.e., then $\mu_k[u_j] \to \mu_k[u]$ weakly as measures.*

Hence, system (S) makes sense even in case u, v are not C^2 but are k-subharmonic, as long as we understand $F_k[u]$ as $\mu_k[u]$.

The case $k = 1$ of system (S) , i.e. $F_1[u] = \Delta u$, and its p-laplacian generalization, have been addressed recently in da Silva and do O (2024), where the authors obtained Brezis-Kamin type estimates Brezis and Kamin (1992) for a quasi-linear system similar to the one being discussed here. In da Silva (2024), we analyzed the Fractional Laplacian equivalent of (S). We plan to generalize this circle of ideas and discuss the fully nonlinear case when $1 < k < \frac{n}{2}$.

For more on the k-Hessian equation see the wonderful monograph Wang (2009).

System (S) above is related to the following system

$$\begin{cases} u = \mathbf{W}_{\frac{2k}{k+1},k+1}\left(v^{q_1}\mathrm{d}\sigma\right), & \mathrm{d}\sigma\text{-a.e in } \mathbb{R}^n, \\ v = \mathbf{W}_{\frac{2k}{k+1},k+1}\left(u^{q_2}\mathrm{d}\sigma\right), & \mathrm{d}\sigma\text{-a.e in } \mathbb{R}^n. \end{cases} \tag{W}$$

Where $\mathbf{W}_{\alpha,p}\mu$ is the Wolff Potential, defined by

$$\mathbf{W}_{\alpha,p}\mu(x) = \int_0^\infty \left(\frac{\mu(B(x,t))}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t}, \quad x \in \mathbb{R}^n. \tag{1}$$

and consequently,

$$\mathbf{W}_{\frac{2k}{k+1},k+1}\mu(x) = \int_0^\infty \left(\frac{\mu(B(x,t))}{t^{n-2k}}\right)^{\frac{1}{k}} \frac{\mathrm{d}t}{t}, \quad x \in \mathbb{R}^n. \tag{2}$$

Notation: For the sake of simplicity will denote $\mathbf{W}_{\frac{2k}{k+1},k+1}\sigma(x)$ by $\mathbf{W}_k\sigma(x)$.

In this paper we will only consider measures $\mu \in M^+(\mathbb{R}^n)$ who satisfy

$$\mathbf{W}_{\alpha,p}\mu(x) < \infty \text{ a.e. in } \mathbb{R}^n \tag{FIN}$$

The following theorem is an adaptation, for our purposes, of (da Silva and do O, 2024, Thm. 1.1).

Theorem 2. *Let $1 \le k < \frac{n}{2}$ and consider $\sigma \in M^+(\mathbb{R}^n)$ satisfying*

$$\sigma(E) \le C_\sigma \operatorname{cap}_{\frac{2k}{k+1},k+1}(E) \quad \textit{for all compact sets } E \subset \mathbb{R}^n, \tag{C}$$

and (FIN), where $\operatorname{cap}_{\alpha,p}(E)$ is the (α,p)-capacity defined by

$$\operatorname{cap}_{\alpha,p}(E) = \inf\{\|f\|_{L^p}^p : f \in L^p(\mathbb{R}^n),\ f \ge 0,\ \mathbf{I}_\alpha f \ge 1 \text{ on } E\}. \tag{3}$$

Then there exists a solution (u,v) to system (W) such that

$$\begin{aligned} c^{-1}\left(\mathbf{W}_k\sigma\right)^{\frac{k(k+q_1)}{k^2-q_1q_2}} &\le u \le c\left(\mathbf{W}_k\sigma + \left(\mathbf{W}_k\sigma\right)^{\frac{k(k+q_1)}{k^2-q_1q_2}}\right), \\ c^{-1}\left(\mathbf{W}_k\sigma\right)^{\frac{k(k+q_2)}{k^2-q_1q_2}} &\le v \le c\left(\mathbf{W}_k\sigma + \left(\mathbf{W}_k\sigma\right)^{\frac{k(k+q_2)}{k^2-q_1q_2}}\right), \end{aligned} \tag{4}$$

where $c = c(n,p,q_1,q_2,\alpha,C_\sigma) > 0$. Furthermore, $u, v \in L^s_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$, for every $s > 0$.

Our main result is the following theorem which is a direct application of the theorem above:

Theorem 3. *Let $1 < k < \frac{n}{2}$ and consider $\sigma \in M^+(\mathbb{R}^n)$ satisfying (C) and (FIN). Then there exists a solution $(u,v) \in \Phi^k(\mathbb{R}^n) \times \Phi^k(\mathbb{R}^n)$ to Syst. (S) such that*

$$\begin{aligned} c^{-1}\left(\mathbf{W}_k\sigma\right)^{\frac{k(k+q_1)}{k^2-q_1q_2}} &\le u \le c\left(\mathbf{W}_k\sigma + \left(\mathbf{W}_k\sigma\right)^{\frac{k(k+q_1)}{k^2-q_1q_2}}\right), \\ c^{-1}\left(\mathbf{W}_k\sigma\right)^{\frac{k(k+q_2)}{k^2-q_1q_2}} &\le v \le c\left(\mathbf{W}_k\sigma + \left(\mathbf{W}_k\sigma\right)^{\frac{k(k+q_2)}{k^2-q_1q_2}}\right), \end{aligned} \tag{5}$$

where $c = c(n,p,q_1,q_2,\alpha,C_\sigma) > 0$. Furthermore, $u, v \in L^s_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$, for every $s > 0$.

In the next theorem we provide a fully nonlinear equivalent result similar to what is presented in Cao and Verbitsky (2017), da Silva and do O (2024).

Theorem 4. *Let $1 < k < \frac{n}{2}$ and consider $\sigma \in M^+(\mathbb{R}^n)$ satisfying (FIN). Then there exists a solution $(u,v) \in \Phi^k(\mathbb{R}^n) \times \Phi^k(\mathbb{R}^n)$ to System (S) satisfying (5) if and only if* *there exists $\lambda > 0$ such that almost everywhere we have*

$$\begin{aligned} \mathbf{W}_k\left((\mathbf{W}_k\sigma)^{q_1\gamma_2}\right) &\leq \lambda\left(\mathbf{W}_k\sigma + (\mathbf{W}_k\sigma)^{\gamma_1}\right) < \infty, \\ \mathbf{W}_k\left((\mathbf{W}_k\sigma)^{q_2\gamma_1}\right) &\leq \lambda\left(\mathbf{W}_k\sigma + (\mathbf{W}_k\sigma)^{\gamma_2}\right) < \infty. \end{aligned} \tag{6}$$

In section 3 we analyze a generalization of System (W), we consider

$$\begin{cases} u = \mathbf{W}_{\alpha,p}\left(v^{q_1}\mathrm{d}\sigma + \mathrm{d}\mu\right), & \mathrm{d}\sigma\text{-a.e in } \mathbb{R}^n, \\ v = \mathbf{W}_{\alpha,p}\left(u^{q_2}\mathrm{d}\sigma + \mathrm{d}\nu\right), & \mathrm{d}\sigma\text{-a.e in } \mathbb{R}^n, \end{cases} \tag{IW}$$

where $\mu, \nu \in M^+(\mathbb{R}^n)$ are given. We aim to prove the following result.
Theorem 5. *Let $\sigma, \mu, \nu \in M^+(\mathbb{R}^n)$ satisfy (C) and (FIN) Then there exists a solution (u, v) to system (IW) such that*

$$\begin{aligned} c^{-1}\left(\mathbf{W}_{\alpha,p}\sigma\right)^{\gamma_1} &\leq u \leq c\left(\mathbf{W}_{\alpha,p}\mu + \mathbf{W}_{\alpha,p}\nu + \mathbf{W}_{\alpha,p}\sigma + \left(\mathbf{W}_{\alpha,p}\sigma\right)^{\gamma_1}\right), \\ c^{-1}\left(\mathbf{W}_{\alpha,p}\sigma\right)^{\gamma_2} &\leq v \leq c\left(\mathbf{W}_{\alpha,p}\mu + \mathbf{W}_{\alpha,p}\nu + \mathbf{W}_{\alpha,p}\sigma + \left(\mathbf{W}_{\alpha,p}\sigma\right)^{\gamma_2}\right), \end{aligned} \tag{7}$$

where $c = c(n, p, q_1, q_2, \alpha, C_\sigma) > 0$. Furthermore,

$$u, v \in L^s_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu) + L^s_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\nu) + L^s_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma),$$

for every $s > 0$.

Organization of the paper

In Secion 2 we give some preliminaries and present the proof of theorems 3 and 4. In Section 3, we discuss generalizations and prove theorem 5. In Section 4, we present our final remarks and some open problems.

Notations and definitions

We assume $\Omega \subseteq \mathbb{R}^n$ is a domain. We denote by $M^+(\Omega)$ the space of all nonnegative locally finite Borel measures on Ω and $\sigma(E) = \int_E d\sigma$ the σ-measure of a measurable set $E \subseteq \Omega$. The letter c or C will always denote a positive constant which may vary from line to line. We understand $F_k[u]$ as the k-Hessian measure $\mu_k[u]$ associated to u, in particular we do not assume $u \in C^2$.

2 Main results

We will need the following result from Phuc and Verbitsky (2008):
Lemma A. *Let $u \geq 0$ be such that $-u \in \Phi_k(\mathbb{R}^n)$, where $1 \leq k < \frac{n}{2}$. If $\mu = \mu_k[-u]$ and $\inf_{\mathbb{R}^n} u = 0$. Then for all $x \in \mathbb{R}^n$,*

$$K^{-1}\mathbf{W}_k\mu(x) \leq u(x) \leq K\mathbf{W}_k\mu(x), \tag{8}$$

for a constant K depending only on n and k.

The following lemma from Cao and Verbitsky (2017) will be needed for the proof of the main theorem.

Lemma B. *Let $\omega \in M^{+}(\mathbb{R}^n)$. For every $r > 0$ and for all $x \in \mathbb{R}^n$, it holds*

$$\mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}\omega)^{r}\mathrm{d}\omega\right)(x) \geq \kappa^{\frac{r}{p-1}}\left(\mathbf{W}_{\alpha,p}\omega(x)\right)^{\frac{r}{p-1}+1}, \tag{9}$$

where κ depends only on n, p.

Proof of Theorem 3

Proof. The proof is similar to the arguments presented in Phuc and Verbitsky (2008),Cao and Verbitsky (2017), da Silva and do O (2024) and uses the idea of successive approximations. Let $(\overline{u}, \overline{v})$ be the solution to the system (W) given by theorem 2. By hypothesis, $\overline{u}(x), \overline{v}(x)$ satisfy estimate (5) , which implies

$$\liminf_{|x|\to\infty} \overline{u}(x) = 0, \quad \liminf_{|x|\to\infty} \overline{v}(x) = 0.$$

Also, notice that $\overline{v}^{q_1}d\sigma, \overline{u}^{q_2}d\sigma \in L^1_{loc}$. This will be important for the uniqueness below. Set

$$\gamma_1 = \frac{k(k+q_1)}{k^2 - q_1 q_2} \quad \text{and} \quad \gamma_2 = \frac{k(k+q_2)}{k^2 - q_1 q_2},$$

and notice that

$$\gamma_1 = \frac{q_1}{k}\gamma_2 + 1 \quad \text{and} \quad \gamma_2 = \frac{q_2}{k}\gamma_1 + 1.$$

Now, define $u_0(x) = \lambda\,(\mathbf{W}_k\sigma)^{\gamma_1}$ and $v_0(x) = \lambda\,(\mathbf{W}_k\sigma)^{\gamma_2}$, where λ is a constant to be chosen. Using lemma B, we obtain:

$$\mathbf{W}_k(v_0^{q_1}d\sigma) = \mathbf{W}_k((\lambda\,(\mathbf{W}_k\sigma)^{\gamma_2})^{q_1}d\sigma) \geq C\,\lambda\,(\mathbf{W}_k\sigma)^{\gamma_1}.$$

By choosing λ suitably small it's possible to obtain $u_0 \leq \mathbf{W}_k(v_0^{q_1}\mathrm{d}\sigma), v_0 \leq \mathbf{W}_k(u_0^{q_2}\mathrm{d}\sigma) \leq \overline{v}$ and

$$u_0 \leq \overline{u},\ v_0 \leq \overline{v}.$$

For $i = 1, 2, \ldots$, let B_i denote the ball centered at the origin of radius i and consider the Dirichlet problem:

$$\begin{cases} F_k[-u_1^i] = \sigma\,v_0^{q_1} \text{ in } B_i, \\ u_1^i = 0 \text{ on } \partial B_i, \end{cases} \tag{D}$$

There is of course an equivalent system associated to $v(x)$, namely, functions v_1^i defined in B_i.

By the existence result (Wang, 2009, Thm. 8.2) and comparison principle (Trudinger and Wang, 2002, Thm. 4.1), system (D) above has a unique solution.

Remark 1. *Notice that we only have uniqueness because $\sigma\,v_0^{q_1}, \sigma\,u_0^{q_2} \in L^1$, which implies in particular continuity with respect to the k-Hessian capacity. It's not known if uniqueness is valid for general μ not integrable.*

Moreover, the sequence of functions $u_1^i(x)$ is increasing, i.e. $u_1^i(x) \leq u_1^{i+1}(x)$, and we claim it is also bounded above. Indeed, by (Phuc and Verbitsky, 2008, Thm. 7.2), the following estimate is true for every i:

$$u_1^i(x) \leq C\mathbf{W}_k v_0^{q_1} \leq C\mathbf{W}_k \overline{v}^{q_1} = \overline{u}$$

Therefore, we conclude that there are k-subharmonic functions $u_1(x), v_1(x)$ such that

$$u_1 = \lim_{i\to\infty} u_1^i, \quad v_1 = \lim_{i\to\infty} v_1^i$$

By the weak continuity of the k-Hessian measure, namely theorem 1, we have (u_1, v_1) satisfy the system

$$\begin{cases} F_k[-u_1] = \sigma\, v_0^{q_1} \text{ in } \mathbb{R}^n, \\ F_k[-v_1] = \sigma\, u_0^{q_2} \text{ in } \mathbb{R}^n. \end{cases} \tag{10}$$

Also, notice that $u_1 \leq \overline{u},\ v_1 \leq \overline{v}$, so we automatically have

$$\liminf_{|x|\to\infty} u_1(x) = \liminf_{|x|\to\infty} v_1(x) = 0.$$

Additionally, it follows that

$$\begin{aligned} u_1 &\geq K^{-1}\mathbf{W}_k(v_0^{q_1} d\sigma) \geq u_0, \\ v_1 &\geq K^{-1}\mathbf{W}_k(u_0^{q_2} d\sigma) \geq v_0. \end{aligned} \tag{11}$$

We apply induction to obtain sequences of functions u_i, v_i such that

$$\begin{cases} F_k[-u_i] = \sigma\, v_{i-1}^{q_1} \quad \text{in} \quad \mathbb{R}^n, \\ F_k[-v_i] = \sigma\, u_{i-1}^{q_2} \quad \text{in} \quad \mathbb{R}^n, \\ u_i \leq \overline{u}, \quad v_i \leq \overline{v} \quad \text{in} \quad \mathbb{R}^n, \\ 0 \leq u_{i-1} \leq u_j, \quad 0 \leq v_{i-1} \leq v_i, \\ \liminf_{|x|\to\infty} u_i(x) = \liminf_{|x|\to\infty} v_i(x) = 0, \end{cases} \tag{12}$$

By the monotone convergence theorem and the weak continuity of the k-Hessian, we conclude that there are functions u, v satisfying

$$\begin{cases} F_k[-u] = \sigma\, v^{q_1} \quad \text{in} \quad \mathbb{R}^n, \\ F_k[-v] = \sigma\, u^{q_2} \quad \text{in} \quad \mathbb{R}^n, \\ \liminf_{|x|\to\infty} u(x) = \liminf_{|x|\to\infty} v(x) = 0, \end{cases} \tag{13}$$

The remaining statements in the theorem follow directly from the fact that

$$u_0 \leq u \leq \overline{u}, \quad v_0 \leq v \leq \overline{v}.$$

and (da Silva and do O, 2024, Lem. 3.4)(alternatively, lemma D below). □

Proof of Theorem 4

Proof.

($\Rightarrow$) Suppose System (S) has a solution $(u,v) \in \Phi^k(\mathbb{R}^n) \times \Phi^k(\mathbb{R}^n)$ satisfying (5), then

$$\begin{aligned} c^{-1}\left(\mathbf{W}_k\sigma\right)^{\gamma_1} \leq u \leq c\left(\mathbf{W}_k\sigma + \left(\mathbf{W}_k\sigma\right)^{\gamma_1}\right), \\ c^{-1}\left(\mathbf{W}_k\sigma\right)^{\gamma_2} \leq v \leq c\left(\mathbf{W}_k\sigma + \left(\mathbf{W}_k\sigma\right)^{\gamma_2}\right), \end{aligned} \tag{14}$$

for some $c > 0$. Now, by lemma A,

$$\begin{aligned} K^{-1}\mathbf{W}_k(v^{q_1}d\sigma) \leq u(x), \\ K^{-1}\mathbf{W}_k(u^{q_2}d\sigma) \leq v(x), \end{aligned} \tag{15}$$

therefore we obtain:

$$\begin{aligned} K^{-1}\mathbf{W}_k(v^{q_1}d\sigma) \geq C\,\mathbf{W}_k\left(\left(\mathbf{W}_k\sigma\right)^{q_1\gamma_2}\mathrm{d}\sigma\right), \\ K^{-1}\mathbf{W}_k(u^{q_2}d\sigma) \geq C\,\mathbf{W}_k\left(\left(\mathbf{W}_k\sigma\right)^{q_2\gamma_1}\mathrm{d}\sigma\right). \end{aligned} \tag{16}$$

If we set $\lambda = \frac{c}{C} > 0$ the result follows.

($\Leftarrow$) This is a direct consequence of (da Silva and do O, 2024, Thm. 1.3), since it give us a solution $(\overline{u},\overline{v})$ to the system (W). If we proceed mutatis mutandis like in the proof of theorem 3 we obtain the desired solution.

$\square$

Remark 2. *In the linear case, $k = 1, u = v, q_1 = q_2 = q$ and σ radially symmetric, we have recently da Silva (2024) given a existence criteria for a weaker criteria than the one above. In particular, it's possible to have solutions that do not satisfy the estimates (5). More precisely, the following example given in Cao and Verbitsky (2016), gives a counter-example:*

$$\sigma(y) = \begin{cases} \frac{1}{|y|^s \log^\beta \frac{1}{|y|}}, & \text{if } |y| < 1/2, \\ 0, & \text{if } |y| \geq 1/2, \end{cases} \tag{17}$$

where $s = (1-q)n + 2q$ and $\beta > 1$. It is unknown at the time of the writing of this paper if it's possible to have solutions without (5) holding when $k > 1$.

3 The inhomogeneous case

In this section we study the case where the complexity of the system is increased due to measures $\mu, \nu \in M^+(\mathbb{R}^n)$. We consider

$$\begin{cases} F_k[-u] = \sigma\, v^{q_1} + \mu, \quad v > 0 \quad \text{in} \quad \mathbb{R}^n, \\ F_k[-v] = \sigma\, u^{q_2} + \nu, \quad u > 0 \quad \text{in} \quad \mathbb{R}^n, \\ \liminf\limits_{|x|\to\infty} u(x) = 0, \quad \liminf\limits_{|x|\to\infty} v(x) = 0. \end{cases} \tag{INH}$$

As before, this system is related to the more general integral system

$$\begin{cases} u = \mathbf{W}_{\alpha,p}\left(v^{q_1}\mathrm{d}\sigma + \mathrm{d}\mu\right), & \mathrm{d}\sigma\text{-a.e in } \mathbb{R}^n, \\ v = \mathbf{W}_{\alpha,p}\left(u^{q_2}\mathrm{d}\sigma + \mathrm{d}\nu\right), & \mathrm{d}\sigma\text{-a.e in } \mathbb{R}^n. \end{cases} \tag{IW}$$

We claim that our results generalize to this scenario as well. First, we need a generalization of lemma B:

Lemma C. *Let $\sigma, \mu \in M^+(\mathbb{R}^n)$. For every $r > 0$ and for all $x \in \mathbb{R}^n$, it holds*

$$\mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}\sigma)^r \mathrm{d}\sigma + \mathrm{d}\mu\right)(x) \geq \kappa^{\frac{r}{p-1}}\left(\mathbf{W}_{\alpha,p}\sigma)\right)^{\frac{r}{p-1}+1}, \tag{18}$$

where κ depends only on n, p.

Proof. The proof is immediate since $(\mathbf{W}_{\alpha,p}\sigma)^r\mathrm{d}\sigma + \mathrm{d}\mu \geq (\mathbf{W}_{\alpha,p}\sigma)^r\mathrm{d}\sigma$. ☐

Like before, the lemma above is the key in showing that a subsolution exists. We recall the following result from (da Silva and do O, 2024, Lem. 3.4):

Lemma D. *Let $\sigma \in M^+(\mathbb{R}^n)$ satisfies (C) and (FIN). Then*

$$\int_{B(x,R)} (\mathbf{W}_{\alpha,p}\sigma)^s \,\mathrm{d}\sigma \leq c(\sigma(B(x,2R) + \sigma(B(x,R)) \tag{19}$$

Proof of Theorem 5

Proof. The method of proof is similar to Cao and Verbitsky (2017); da Silva and do O (2024); da Silva (2024), namely use sub-super solutions. Even though it's not entirely obvious that the same method should work due to the interactions between σ, μ, ν, as pointed out by (da Silva and do O, 2024, Sec. 5), it turns out that with some adjustments the proof is still valid. We recall the following notation:

$$\gamma_1 = \frac{(p-1)(p-1+q_1)}{(p-1)^2 - q_1q_2} \quad \text{and} \quad \gamma_2 = \frac{(p-1)(p-1+q_2)}{(p-1)^2 - q_1q_2},$$

Notice that by definition

$$\gamma_1 = \frac{q_1}{p-1}\gamma_2 + 1 \quad \text{and} \quad \gamma_2 = \frac{q_2}{p-1}\gamma_1 + 1.$$

The idea to find a subsolution is to use lemma C. Fix $\lambda > 0$, to be defined later and define

$$(\underline{u}, \underline{v}) = \left(\lambda(\mathbf{W}_{\alpha,p}\sigma)^{\gamma_1}, \lambda(\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2}\right)$$

Then we obtain

$$\begin{aligned} \mathbf{W}_{\alpha,p}(\underline{v}^{q_1}\mathrm{d}\sigma + \mathrm{d}\mu) &= \lambda^{\frac{q_1}{p-1}}\mathbf{W}_{\alpha,p}((\mathbf{W}_{\alpha,p}\sigma)^{q_1\gamma_2}\mathrm{d}\sigma + \mathrm{d}\mu) \geq \lambda^{\frac{q_1}{p-1}}C(\mathbf{W}_{\alpha,p}\sigma)^{\frac{q_1}{p-1}\gamma_2+1} \\ &= \lambda^{\frac{q_1}{p-1}}C(\mathbf{W}_{\alpha,p}\sigma)^{\gamma_1}, \\ \mathbf{W}_{\alpha,p}(\underline{u}^{q_2}\mathrm{d}\sigma + \mathrm{d}\nu) &= \lambda^{\frac{q_2}{p-1}}\mathbf{W}_{\alpha,p}((\mathbf{W}_{\alpha,p}\sigma)^{q_2\gamma_1}\mathrm{d}\sigma + \mathrm{d}\nu) \geq \lambda^{\frac{q_2}{p-1}}C(\mathbf{W}_{\alpha,p}\sigma)^{\frac{q_2}{p-1}\gamma_1+1} \end{aligned}$$

$$= \lambda^{\frac{q_2}{p-1}} C(\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2}.$$

By choosing λ sufficiently small, we can guarantee that

$$\underline{u} \leq \mathbf{W}_{\alpha,p}(\underline{v}^{q_1}\mathrm{d}\sigma + \mathrm{d}\mu),$$
$$\underline{v} \leq \mathbf{W}_{\alpha,p}(\underline{u}^{q_2}\mathrm{d}\sigma + \mathrm{d}\nu),$$

which confirms that $(\underline{u}, \underline{v})$ is a subsolution.

The goal now is to find a supersolution. Let's now define

$$(\overline{u}, \overline{v}) = \left(\lambda\left(\mathbf{W}_{\alpha,p}\mu + \mathbf{W}_{\alpha,p}\nu + \mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_1}\right), \lambda\left(\mathbf{W}_{\alpha,p}\mu + \mathbf{W}_{\alpha,p}\nu + \mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2}\right)\right)$$

where λ is a constant to be determined later. Then we have

$$\begin{aligned}
\mathbf{W}_{\alpha,p}(\overline{v}^{q_1}\mathrm{d}\sigma + \mathrm{d}\mu)(x) &\leq \lambda^{\frac{q_1}{p-1}} C \int_0^\infty \left(\frac{\int_{B(x,t)}(\mathbf{W}_{\alpha,p}\sigma)^{q_1} + (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2 q_1}\mathrm{d}\sigma + \mu(B(x,t))}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} \\
&\leq \lambda^{\frac{q_1}{p-1}} C \Big[\int_0^\infty \left(\frac{\int_{B(x,t)}(\mathbf{W}_{\alpha,p}\mu)^{q_1}\mathrm{d}\sigma}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} \\
&+ \int_0^\infty \left(\frac{\int_{B(x,t)}(\mathbf{W}_{\alpha,p}\nu)^{q_1}\mathrm{d}\sigma}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} \\
&+ \int_0^\infty \left(\frac{\int_{B(x,t)}(\mathbf{W}_{\alpha,p}\sigma)^{q_1}\mathrm{d}\sigma}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} \\
&+ \int_0^\infty \left(\frac{\int_{B(x,t)}(\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2 q_1}\mathrm{d}\sigma}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} \\
&+ \mathbf{W}_{\alpha,p}\mu\Big] \\
&= \lambda^{\frac{q_1}{p-1}} C[I + II + III + IV + \mathbf{W}_{\alpha,p}\mu], \qquad (20)
\end{aligned}$$

It's enough to estimate III and IV, since I, II are of type III.

By lemma D and (da Silva and do O, 2024, Proof of thm. 1.1) we have:

$$\begin{aligned}
III &\leq \int_0^\infty \left(\frac{c\sigma(B(x,2t)) + c\sigma(B(x,t))}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} \\
&\leq c \int_0^\infty \left(\frac{\sigma(B(x,2t))}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} + c\,\mathbf{W}_{\alpha,p}\sigma(x) \\
&\leq c\, 2^{\frac{n-\alpha p}{p-1}} \int_0^\infty \left(\frac{\sigma(B(x,t))}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} + c\,\mathbf{W}_{\alpha,p}\sigma(x) \\
&\leq c\mathbf{W}_{\alpha,p}\sigma(x), \qquad (21)
\end{aligned}$$

and

$$IV \leq c\Big[\mathbf{W}_{\alpha,p}\sigma(x) + (\mathbf{W}_{\alpha,p}\sigma(x))^{\frac{q_1}{p-1}\gamma_2 + 1}\Big].$$

Combining everything together we have:

$$\mathbf{W}_{\alpha,p}(\overline{v}^{q_1}\mathrm{d}\sigma+\mathrm{d}\mu)(x) \leq \lambda^{\frac{q_1}{p-1}}C[\mathbf{W}_{\alpha,p}\nu(x)+\mathbf{W}_{\alpha,p}\sigma(x)+(\mathbf{W}_{\alpha,p}\sigma(x))^{\frac{q_1}{p-1}\gamma_2+1}+\mathbf{W}_{\alpha,p}\mu(x)]$$

Likewise, by symmetry we also have

$$\mathbf{W}_{\alpha,p}(\overline{u}^{q_2}\mathrm{d}\sigma+\mathrm{d}\nu)(x) \leq \lambda^{\frac{q_2}{p-1}}C[\mathbf{W}_{\alpha,p}\mu(x)+\mathbf{W}_{\alpha,p}\sigma(x)+(\mathbf{W}_{\alpha,p}\sigma(x))^{\frac{q_2}{p-1}\gamma_1+1}+\mathbf{W}_{\alpha,p}\nu(x)]$$

By choosing $\lambda > 0$ large enough we guarantee that $(\overline{u}, \overline{v})$ is a supersolution. Using standard iteration arguments, we conclude that there is a solution (u, v) to system (IW) satisfying

$$\underline{u} \leq u \leq \overline{u}$$
$$\underline{v} \leq v \leq \overline{v}$$

Since by (da Silva and do O, 2024, Lem. 3.4), $\overline{u}, \overline{v} \in L^s_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu) + L^s_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\nu) + L^s_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$, we automatically have $u, v \in L^s_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)+L^s_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\nu)+L^s_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$ □

With theorem 5 proved, one can easily prove existence results to systems of the form (INH). Also, the following systems:

$$\begin{cases} -\Delta_p u = \sigma\, v^{q_1} + \mu, & u > 0 \quad \text{in} \quad \mathbb{R}^n, \\ -\Delta_p v = \sigma\, u^{q_2} + \nu, & v > 0 \quad \text{in} \quad \mathbb{R}^n, \\ \varliminf\limits_{|x|\to\infty} u(x) = 0, & \varliminf\limits_{|x|\to\infty} v(x) = 0, \end{cases} \tag{22}$$

and

$$\begin{cases} (-\Delta)^\alpha u = \sigma\, v^{q_1} + \mu, & u > 0 \quad \text{in} \quad \mathbb{R}^n, \\ (-\Delta)^\alpha v = \sigma\, u^{q_2} + \nu, & v > 0 \quad \text{in} \quad \mathbb{R}^n, \\ \varliminf\limits_{|x|\to\infty} u(x) = 0, & \varliminf\limits_{|x|\to\infty} v(x) = 0. \end{cases} \tag{23}$$

can be analyzed in a very similar way, since they are both connected to (IW).

4 Final remarks and open questions

The results above lead to these natural questions:

1. In these notes we have assumed $k < \frac{n}{2}$ because the Wolff potential $\mathbf{W}_{\alpha,p}\sigma$ estimates requires $0 < \alpha < \frac{n}{p}$, which in our case is the same thing of requiring $k < \frac{n}{2}$. We may then ask what happens if $k > \frac{n}{2}$, namely, do we still have any sort of Brezis-Kamin estimates (5) in case we have non trivial solutions? A particular case would

be the system below not covered in this article:

$$
\begin{cases}
\det[D^2u] = \sigma\, v^{q_1}, \quad v > 0 \quad \text{in} \quad \mathbb{R}^n, \\
\det[D^2v] = \sigma\, u^{q_2}, \quad u > 0 \quad \text{in} \quad \mathbb{R}^n, \\
\liminf\limits_{|x|\to\infty} u(x) = 0, \quad \liminf\limits_{|x|\to\infty} v(x) = 0, \\
\text{u and v convex.}
\end{cases}
\tag{24}
$$

Labutin's local estimates Labutin (2002) are still valid, so it may be possible to still obtain solutions satisfying (5) but using different methods of the ones presented here.

2. We haven't discuss here the regularity of the solutions, but we believe some regularity theory could be stablished based on the regularity theory which already exists for equations involving the k-Hessian. For example, by using the same type of arguments as the ones presented in Wang (2009) and Chou and Wang (2001). In particular, it's possible but not clear if Holder regularity could a priori be obtained for (S), depending on the values of k, n, q_1, q_2.

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