Exam III

1. Show that $(X^\circ)^\circ = X^\circ$, where X° is the set of all interior points of X. Conclude that X° is an open set.

   Solution. We obviously have $(X^\circ)^\circ \subseteq X^\circ$. Conversely, take $a \in X^\circ$, then there exists $\delta > 0$ such that $(a-\delta, a+\delta) \subseteq X$. By definition, every point in $(a-\delta, a+\delta)$ is itself an interior point, hence $(a-\delta, a+\delta) \subseteq X^\circ$ and it follows that $a \in (X^\circ)^\circ$. $\square$

2. Given a point $a \in X \subseteq \mathbb{R}$. We call a a boundary point of X if for every open interval I containing a, we have $I \cap X \neq \emptyset$ and $I \cap (\mathbb{R} - X) \neq \emptyset$. The set of all boundary points, ∂X, is called the *boundary* of X. Show that a set A is open if and ony if $A \cap \partial A = \emptyset$.

   Solution. Notice that if a point is an interior point then it's not a boundary point. Hence, if A contains only interior points, it doesn't contain boundary points. Conversely, if $A \cap \partial A = \emptyset$. Then given a point $a \in A$, we have $a \notin \partial A$, so there should be a neighborhood of a entire contained in A, thus a is an interior point. $\square$

3. Let $X \subseteq \mathbb{R}$. Show that $\overline{X} = X \cup \partial X$. Conclude that X is closed if and only if $\partial X \subseteq X$.

   Solution. It follows directly from the definitions that $X \cup \partial X \subseteq \overline{X}$. Conversely, take $a \in \overline{X}$ and suppose that $a \notin \partial X$, that is to say, there is a neighborhood of a containing only points inside X or only points outside X, but since $a \in \overline{X}$, it must be points inside X, i.e. a is an interior point, in particular, it's in X, hence $\overline{X} \subseteq X \cup \partial X$. $\square$

4. Show that finite union of compact sets is compact, and also arbitrary intersection of compact sets is compact as well.

   Solution. By Lebesgue Theorem, a compact set is closed and bounded, hence a finite union of compact sets is still closed and bounded, hence compact. Same argument is valid for arbitrary intersections. $\square$

5. Show that every compact set X, satisfying $X' = \emptyset$, is finite. Where X' is the derived set of X, i.e. the collection of all accumulation points.

   Solution. By hypothesis, X has only isolated points. Cover each isolated point by a open set containing only that point, since X is compact, it has a finite subcover, hence the number of isolated points are finite, so X is finite since all of its points are isolated. $\square$

Extra (2 pts). Show that the sum of the lengths of removed intervals in the construction of the Cantor set is one. Where by length of (a, b), I mean $b - a$. *[Hint: The formula for the sum of terms of a geometric progression $a, ar, ar^2, \ldots$ with $|r| < 1$ is $\frac{a}{1-r}$]*

Solution. $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \ldots = \frac{1/3}{1-2/3} = 1$ $\square$