Exam I

Choose (only) 4 questions:

1. Give an example of three functions f, g, h such that $f \circ (g + h) \neq f \circ g + f \circ h$
2. Find the largest natural number m such that $n^3 - n$ is divisible by m for all $n \in \mathbb{N}$. Prove your assertion.
3. Let $n \in \mathbb{N}$. Show that there is no $m \in \mathbb{N}$ such that $n < m < n + 1$.
4. Prove the induction principle assuming the principle of well-ordering.
5. Show that the set $P = \{\, n \in \mathbb{N} \,;\, n \text{ is prime} \,\}$ is infinite.
6. Let Y be countable and $f : X \to Y$ such that $f^{-1}(y)$ is countable for each $y \in Y$. Show that X is countable.
7. Given an example of $X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots$, an infinite sequence of nested **infinite** subsets such that
$$\bigcap_{i=1}^{\infty} X_i = \emptyset$$
8. Show that the set of all finite subsets of $\mathbb{N}$ is countable.