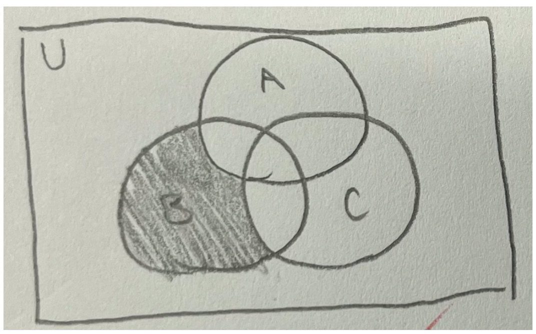
Exam I

Choose (only) 5 questions:

1. Find the cardinality of the the set $\{\{1\}, 3, \{\{1\}, 3\}\}$

   Proof. The set has 3 elements: $\{1\}, 3$ and $\{\{1\}, 3\}$. □

2. For sets A, B and C, and a universal set U, draw the Venn diagram representing the following set $A^c \cap (B - C)$

   Proof. □

3. Let $\mathcal{P}(A)$ denotes the power set of A. Show that

$$\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$$

   Proof. A set X is a subset of $A \cap B$ if and only if it is a subset of A and also a subset of B. □

4. Using induction prove that

$$\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

   Proof. If $n = 1$, then $\frac{1}{2!} = 1 - \frac{1}{2!}$. Suppose the result true for $n = k$. Take $n = k + 1$, then:

$$\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!}$$

   Since

$$1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+2)!},$$

   we conclude that

$$\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+2)!},$$

   and the result follows by induction. □

5. Using induction prove that if a set A has n elements then the power set $\mathcal{P}(A)$ has 2^n elements.

   Proof. If $n = 1$, then A has only two subsets, namely itself and $\emptyset$. Suppose the result valid for $n = k$ and take a set A with $k + 1$ elements. Choose one element $x \in A$, we can divide the power set in two sets, one containing all subsets that don't have 'x' and the other all subsets containing x. By hypothesis, in both cases we have 2^k elements, hence the power set has $2^k + 2^k$ elements, i.e. 2^{k+1}. □

6. Rewrite the following sentence to be of the form "If P, then Q.": *An integer is even provided it is not odd.* Make sure your new wording does not change its meaning.

   Proof. If an integer is not odd then it is even. □

7. Negate the following sentence: *If I don't pass Intro to Proofs and Linear Algebra this semester, than I will take Real Analysis and Abstract algebra next semester.*

   Proof. I don't pass Intro to Proofs and Linear Algebra this semester and I will not take Real Analysis or Abstract algebra next semester. □

8. Translate the following to plain English, and then write down whether it is true or false:

   $$\exists n \in \mathbb{N} : \forall y \in \mathbb{Z}, 0 \leq y^n$$

   Proof. There exists an natural number n such that for every integer y we have 0 less than or equal to y^n.

   This is a true statement, $n = 2$ satisfies the condition, for example. □