Exam I

Choose (only) 5 questions:

1. Assume that n is a positive integer. Prove that if one selects any $n+1$ numbers from the $\{1, 2, \ldots, 2n\}$, then two of the selected numbers will sum to $2n+1$.

   Proof. To each number k we associate a unique ball. Label n boxes with labels $\{1, 2n\}$, $\{2, 2n-1\}$,...,$\{n, n+1\}$, and put ball k in the box $\{k, 2n+1-k\}$. By the pigeonhole principle, at least one box will contain 2 balls. □

2. Find your own real-world example of the pigeonhole principle.

   Proof. In a group of 13 people, two of them share the same birthday month. □

3. Give an example of 100 numbers from $\{1, 2, \ldots, 200\}$ where not one of your numbers divides another.

   Proof. $200, 199, 198, \ldots, 101$. □

4. Prove that any set of seven integers contains a pair whose sum or difference is divisible by 10.

   Proof. For each number $0, 1, \ldots, 9$ we associate a unique ball. Label 6 boxes with labels $\{0\}$, $\{5\}$,$\{1, 9\}$,$\{2, 8\}$,$\{3, 7\}$,$\{4, 6\}$. For each number of the given seven, we look at the last digit, and put a ball in the box containing that digit. Since we have only 6 boxes, the pigeonhole principle implies that at least one box will contain 2 balls. □

5. If m and n are integers **with same parity**, then $7m - 3n$ is even.

   Proof. If $m = 2p, n = 2q$, then $7m - 3n = 14p - 6q = 2k$ is even. Similarly, if $m = 2p+1, n = 2q+1$, then $7m - 3n = 14p + 7 - 6q - 3 = 2k$ is also even. □

6. Prove that for every integer n, either $n^2 \equiv 0 \pmod 4$ or $n^2 \equiv 1 \pmod 4$

   Proof. If $\overline{n} = \overline{2p}$, then $\overline{n^2} = \overline{4p^2} = \overline{0}$. If $\overline{n} = \overline{2p+1}$, then $\overline{n^2} = \overline{4p^2 + 4p + 1} = \overline{0} + \overline{0} + \overline{1} = \overline{1}$. □

7. Prove that n is even if and only if n^2 is even. (You'll have to prove two things: if n is even then n^2 is even; and conversely, if n^2 is even then n is even.)

   Proof. Suppose $n = 2k$ then $n^2 = 4k^2 = 2l$ is obviously even. Conversely, suppose n^2 is even. Since 2 divides n^2 then 2 divides n, hence n is even. □

8. Show that the following conjecture is false by finding a counter-example: *Let* $f(n) = n^2 - n + 5$. *For any natural number* $n \in \mathbb{N}$, $f(n)$ *is prime.*

   Proof. $f(5) = 25 - 5 + 5 = 25$ is not prime. □