Exercises

17. Show that for every $n \in \mathbb{N}$, $0 < e - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}\right) < \frac{1}{n!n}$. Conclude that $e \notin \mathbb{Q}$.

    Solution. Recall that $e = \sum\limits_{n=0}^{\infty} \frac{1}{n!}$ is a sum of positive terms. In particular, the partial sum $s_n = \sum\limits_{k=0}^{n} \frac{1}{k!}$ is increasing, so $s_n < e \Rightarrow e - s_n > 0$. On the other hand, $e - s_n = \sum\limits_{k=n+1}^{\infty} \frac{1}{k!} = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \ldots < \frac{1}{(n+1)!} + \frac{1}{(n+1)(n+1)!} + \frac{1}{(n+1)^2(n+1)!} + \ldots$. The last series is the geometric series $\frac{1}{(n+1)!} \sum\limits_{k=0}^{\infty} \frac{1}{(n+1)^k} = \frac{1}{(n+1)!} \frac{(n+1)}{n} = \frac{1}{n!n}$. □

21. Let $p(x) \in \mathbb{R}[x]$ be a polynomial of degree 2 or more. Show that the series $\sum \frac{1}{p(n)}$ converges.

    Solution. Let $p(x) = a_k x^k + \ldots + a_1 x + a_0$, with $k \geq 2$. For n sufficiently large we have $|a_k n^k| > n^2$, hence
    $$\left|\frac{1}{p(n)}\right| < \frac{1}{n^2}.$$
    By comparison, $\sum \frac{1}{p(n)}$ converges. □

23. Let $a \in \mathbb{R}$. Show that the series $\sum\limits_{n=0}^{\infty} \frac{a^2}{(1+a^2)^n}$ converges and find its sum.

    Solution. Apply the root test: $\lim \sqrt[n]{\frac{a^2}{(1+a^2)^n}} = \frac{1}{1+a^2} < 1$, hence the series absolutely converges. It is a geometric series, so $\sum\limits_{n=0}^{\infty} \frac{a^2}{(1+a^2)^n} = a^2 \sum\limits_{n=0}^{\infty} \frac{1}{(1+a^2)^n} = 1 + a^2$. □

29. Show that the set of accumulation points of the sequence $x_n = \cos n$ is the closed interval $[-1, 1]$.

    Solution. Notice that $\cos(n)$ is not periodic if $n \in \mathbb{N}$, since the period of $\cos(x)$ is 2π, if there was n, m such that $\cos(n) = \cos(m)$ then $n - m = 2k\pi$, and 2π would be a natural number, a contradiction. Hence, $\cos(n)$ takes different values in $[-1, 1]$.

    The difficulty here is to show that all numbers of $[-1, 1]$ are close to this countable subset. That is to say, given $x \in [-1, 1]$ and $\epsilon > 0$ we can find infinitely many $n \in \mathbb{N}$ such that $|x - \cos(n)| < \epsilon$. Equivalently, given $x \in [0, 2\pi]$ we can find $y \in \mathbb{R}$ such that $n = y + 2k\pi$ and $|x - y| < \epsilon$. Instead of that, we prove that stronger fact that the set $\{n + 2k\pi; n \in \mathbb{N}, k \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Indeed, given $x \in \mathbb{R}$, suppose $x > 0$, and write x as the sum $x = \lfloor x \rfloor + \{x\}$, its integral and fractional part. The result follows then if we can prove that $\{2k\pi\}$ is dense in $[0, 1]$. Given any $\epsilon > 0$, divide $[0, 1]$ in intervals of length ϵ, then necessarily two numbers of the form $\{2p\pi\}$ and $\{2q\pi\}$ will be in the same interval. Suppose $p > q$, then $2(p - q)\pi$ will be in the first or last interval, hence multiples of $2(p - q)\pi$ will be in all intervals. □

30. Let $a_1 \geq a_2 \geq \ldots \geq 0$ and $s_n = a_1 - a_2 + \ldots + (-1)^{n-1} a_n$. Show that s_n is bounded and

$$\limsup s_n - \liminf s_n = \lim a_n$$

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Solution. Notice that a_n is bounded and monotone, hence converges and any subsequence of it will also converge. Also, $s_{2n} = (a_1 - a_2) + (a_3 - a_4) + \ldots + (a_{2n-1} - a_{2n})$, and since a_n is nonincreasing, it follows that s_{2n} is nondecreasing and bounded by a_1. Similarly, s_{2n-1} is nonincreasing and bounded by a_1. Both subsequences are monotone and bounded hence converge and $s_{2n-1} > s_{2n}$, so one converges to the greatest accumulation point and the other necessarily converges to the smallest one. Moreover $s_{2n-1} - s_{2n} = a_{2n}$, taking the limit we obtain

$$\lim(s_{2n-1} - s_{2n}) = \limsup s_n - \liminf s_n = \lim a_{2n} = \lim a_n$$

$\square$