Exercises

3. Given $A, B \subseteq E$, show that $A \subseteq B$ if and only if $A \cap B^c = \emptyset$.

   Solution. Suppose $A \subseteq B$. Then $x \in A \iff x \in B \iff x \notin B^c$, hence $A \cap B^c = \emptyset$. Conversely, if $A \cap B^c = \emptyset$. Take $x \in A$, then $x \notin B^c \iff x \in B$, we conclude that $A \subseteq B$. □

4. Give examples of sets A, B, C such that $(A \cup B) \cap C \neq A \cup (B \cap C)$.

   Solution. Take $A = \{1\}, B, C = \emptyset$, then $(A \cup B) \cap C = \emptyset$, but $A \cup (B \cap C) = \{1\}$ □

8. Show that a function $f : A \to B$ is injective if and only if $f(A - X) = f(A) - f(X)$ for every $X \subseteq A$.

   Solution. Suppose f injective. Take $y \in f(A - X)$,i.e. $y = f(a)$ for $a \in A$, but $a \notin X$, by the injectivity of f, $y \in f(A) - f(X)$. Now, suppose $y \in f(A) - f(X)$, then $y = f(a)$ but $y \neq f(x)$ for any $x \in X$, again, by the injectivity of f, $a \notin X$, hence $y \in f(A - X)$.

   Conversely, suppose $f(A - X) = f(A) - f(X)$. Set $X = \{a\}$ for $a \in A$, then $f(A - \{a\}) = f(A) - f(a)$. In particulat, if $b \neq a$ then $f(b) \neq f(a)$. □

9. Let $f : A \to B$ be given. Show that

   a. For every $Z \subseteq B$, we have $f(f^{-1}(Z)) \subseteq Z$.

   b. $f(x)$ is surjective if and only if $f(f^{-1}(Z)) = Z$ for every $Z \subseteq B$.

   Solution. a. Take $y \in f(f^{-1}(Z))$, then $y = f(x)$, with $x \in f^{-1}(Z)$, that is , $f(x) = z$, for some $z \in Z$. Then $y \in Z$.

   b. Suppose f surjective. Take $z \in Z$, then $\exists x \in X$ such that $z = f(x)$ and $x \in f^{-1}(Z)$ by definition. Hence, $Z \subseteq f(f^{-1}(Z))$ and equality follows by letter a. Conversely, suppose $f(f^{-1}(Z)) = Z$. Take $Z = B$ then $f(A) = B$.

   □

12. Let $\mathcal{F}(X; Y)$ denote the set of all functions with domain X and codomain Y. Given the sets A, B, C, show that there is a bijection

$$\mathcal{F}(A \times B; C) \to \mathcal{F}(A; \mathcal{F}(B; C)).$$

   Solution. Let $f : A \times B \to C$ be given and for each $a \in A$ define a function $f_a : B \to C$ given by $f_a(b) = f(a, b)$. Consider the function $\bar{f} : A \to \mathcal{F}(B; C)$ given by $\bar{f}(a) = f_a$. Then the correspondence $f \mapsto \bar{f}$ is the required bijection. Given any function $h : A \to \mathcal{F}(B; C)$, let $h_a = h(a)$ and set $f(a, b) = h_a(b)$ then $\bar{f} = h$, hence $f \mapsto \bar{f}$ is surjective. Injectivity is immediate by construction. □