Proof Assume, to the contrary, that none of $a+b$, $a+c$ and $b+c$ are irrational. Then $p=a+b$, $q=a+c$ and $r=b+c$ are all rational. Therefore, $p+q+r=2a+2b+2c$ is rational. Since $2r=2b+2c$ is rational, $(p+q+r)-2r=2a$ is rational, as is $2a/2=a$. Similarly, b and c are rational and so all of a, b and c are rational. This is a contradiction. ■

Proof Evaluation The proposed proof attempts to prove the given statement using a proof by contradiction. However, there are logical errors here. In a proof by contradiction, it should be assumed, to the contrary, that at least one of $a+b$, $a+c$ and $b+c$ is irrational but none of a, b and c are irrational. This would then imply that all of a, b and c are rational, from which it can be shown that all of $a+b$, $a+c$ and $b+c$ are rational. This would be a contradiction. ♦

EXERCISES FOR CHAPTER 7

7.1. We saw in Result 4.8 for integers a and b that $3 \mid ab$ if and only if $3 \mid a$ or $3 \mid b$. Use this fact to prove that if m is an integer such that $10 \mid m$ and $12 \mid m$, then $60 \mid m$.

7.2. Let $a, b, m \in \mathbf{Z}$ with $m \geq 2$ such that $a \equiv b \pmod{m}$.

(a) According to Result 4.11, if $c, d \in \mathbf{Z}$ such that $c \equiv d \pmod{m}$, then $ac \equiv bd \pmod{m}$. Show that $a^2 \equiv b^2 \pmod{m}$ and $a^3 \equiv b^3 \pmod{m}$.

(b) Is it true that $a^2 \equiv b \pmod{m}$?

(c) Prove that $a^n \equiv b^n \pmod{m}$ for every positive integer n.

(d) Use (c) to prove that $8 \mid (3^{2n}-1)$ for every positive integer n.

(e) According to Result 4.6(b), if x is an odd integer, then $8 \mid (x^2-1)$. Use this fact to prove that $8 \mid (3^{2n}-1)$ for every positive integer n.

7.3. (a) Let $m \in \mathbf{Z}$. Prove that if m is the product of four consecutive integers, then $m+1$ is a perfect square (that is, $m+1=k^2$ for some $k \in \mathbf{Z}$).

(b) Prove, for every positive integer n, that neither $n(n+1)$ nor $n(n+2)$ is a perfect square.

(c) Prove that the product of three consecutive integers is always divisible by 6 but not always divisible by 9. When will it be divisible by 12?

7.4. Let $a, b \in \mathbf{N}$. Prove that if $a+b$ is even, then there exist nonnegative integers x and y such that $x^2-y^2=ab$.

7.5. It follows from Result 4.6(b) that if a is an odd integer, then $a^2 \equiv 1 \pmod{8}$. Use this fact to prove that if b is an odd integer, then $b^{2^n} \equiv 1 \pmod{2^{n+2}}$ for every positive integer n.

7.6. We saw in Exercise 4.90 that

$$\text{If } a, b, c, d \in \mathbf{R}^+ \text{ such that } a \geq b \text{ and } c \geq d, \text{ then } ac \geq bd. \tag{7.9}$$

(a) Use (7.9) to prove that if $a, b \in \mathbf{R}^+$ such that $a \geq b$, then $\sqrt{a} \geq \sqrt{b}$.

(b) Without using (7.9), prove that if $a, b \in \mathbf{R}^+$ such that $a \geq b$, then $\sqrt{a} \geq \sqrt{b}$.

7.7. (a) Let $m=2k$ be an even integer where $k \in \mathbf{Z}$. Prove that if a and b are integers such that $a+b \geq m$, then either $a \geq k$ or $b \geq k+1$.

(b) Let $m=3k$ for some $k \in \mathbf{N}$. Prove that if $a, b, c \in \mathbf{N}$ such that $a+b+c \geq m$, then $a \geq k$, $b \geq k$ or $c \geq k+2$.

(c) A set S consists of 20 positive integers whose sum is an even integer. Prove that at least 4 elements of S are congruent to 0 modulo 4, at least 5 are congruent to 1 modulo 4, at least 7 are congruent to 2 modulo 4 or at least 8 are congruent to 3 modulo 4.

7.8. (a) Express the following statement in words: $\forall n \in \{3, 4, 5, \ldots\}, \exists a_1, a_2, \ldots, a_n \in \mathbf{N}$ with $a_1 < a_2 < \cdots < a_n$ such that $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 1$.
(b) Express in words the negation of the statement in (a).
(c) One of the statements in (a) and (b) is true. Determine, with proof, which is true.

7.9. Prove for every positive integer m that is a multiple of 8, there exist two positive integers a and b that differ by m such that ab is a perfect square.

7.10. A sequence $\{a_n\}$ is defined recursively by $a_1 = 7$ and $a_n = 4a_{n-1} - 9$ for $n \geq 2$. Conjecture a formula for a_n and verify your conjecture for every positive integer n.

7.11. The triples (3, 4, 5), (5, 12, 13) and (7, 24, 25) are called Pythagorean triples because $3^2 + 4^2 = 5^2$, $5^2 + 12^2 = 13^2$ and $7^2 + 24^2 = 25^2$. There are infinitely many Pythagorean triples.

(a) Prove for every odd integer $a \geq 3$ that there exists an even integer b such that $(a, b, b + 1)$ is a Pythagorean triple.
(b) Prove for every odd integer $a \geq 3$ and positive integer n, that there exist n positive even integers $b_1, b_2, \ldots, b_n$ such that $a^2 + b_1^2 + b_2^2 + \cdots + b_n^2 = c^2$ for some positive integer c.

7.12. Use induction to prove that $11^n \equiv 1 \pmod 8$ or $11^n \equiv 3 \pmod 8$ for every nonnegative integer n.

7.13. Prove for every three integers a, b and c that an even number of the integers $a + b$, $a + c$ and $b + c$ are odd.

7.14. Evaluate the proof of the following statement.

Statement If x is an integer such that $3 \mid (x - 5)$, then $3 \mid (7x - 2)$.

Proof The integer $x = 5$ has the property that $3 \mid (x - 5)$. Furthermore, for $x = 5$, $3 \mid (7x - 2)$. ■

7.15. Evaluate the proof of the following statement.

Statement Let $x, y, z \in \mathbf{Z}$ such that $3x + 5y = 7z$. If at least one of x, y and z is odd, then at least one of x, y, z is even.

Proof Let $x, y, z \in \mathbf{Z}$ such that $3x + 5y = 7z$. Assume, to the contrary, that none of x, y and z is odd and that none of x, y and z is even. This is impossible. ■

7.16. Evaluate the proof of the following statement.

Statement A sequence $\{a_n\}$ of integers is defined recursively by $a_1 = 1$, $a_2 = 3$, $a_3 = 6$ and $a_n = a_{n-1} + 3a_{n-2} + 6a_{n-3}$ for $n \geq 4$. Then $3 \mid a_n$ for every integer $n \geq 2$.

Proof We proceed by induction. Since $a_2 = 3$, it follows that $3 \mid a_k$ for $k = 2$. Assume that $3 \mid a_k$ for an integer $k \geq 2$. Thus, $a_k = 3x$ for some integer x. We show that $3 \mid a_{k+1}$. Now

$$\begin{aligned} a_{k+1} &= a_k + 3a_{k-1} + 6a_{k-2} = 3x + 3a_{k-1} + 6a_{k-2} \\ &= 3(x + a_{k-1} + 2a_{k-2}). \end{aligned}$$

Since $x + a_{k-1} + 2a_{k-2}$ is an integer, $3 \mid a_{k+1}$. By the Principle of Mathematical Induction, $3 \mid a_n$ for every integer $n \geq 2$. ■

7.17. Evaluate the proof of the following statement.

Statement Let $a \in \mathbf{R}^+$ and let $S = \{2^r : r \in \mathbf{Q}\}$. If $a \notin S$, then $\log_2 a$ is irrational.

Proof Assume, to the contrary, that $\log_2 a$ is rational. Then $\log_2 a = b \in \mathbf{Q}$ and so $a = 2^b$. Since $b \in \mathbf{Q}$, it follows that $a \in S$, which is a contradiction. ■

7.18. Evaluate the proof of the following statement.

Statement Let $a, b, c \in \mathbf{Z}$. If all of the integers $3a + 4b$, $5b + 6c$ and $7c + 8a$ are odd, then all of a, b, c are odd.

Proof Assume, to the contrary, that not all of the integers $3a + 4b$, $5b + 6c$ and $7c + 8a$ are odd, say $3a + 4b$ is not odd. Then $3a + 4b$ is even and so $3a + 4b = 2d$ for some integer d. Hence, $3a = 2d - 4b = 2(d - 2b)$. Since $d - 2b \in \mathbf{Z}$, it follows that $3a$ is even. This, however, implies that a is even and so not all of a, b, c are odd. ■

7.19. Evaluate the proof of the following statement.

Statement Let $a, b, c \in \mathbf{Z}$. Then $ab + ac + bc$ is even if and only if at most one of a, b and c is odd.

Proof We consider the following cases.

Case 1. *None of* a, b *and* c *is odd.* Then all of a, b and c are even. Hence, ab, ac and bc are even, as is $ab + ac + bc$.

Case 2. *Exactly one of* a, b *and* c *is odd, say* a *is odd*. Then b and c are even. Hence, all of ab, ac and bc are even, as is $ab + ac + bc$.

Case 3. *Exactly two of* a, b *and* c *are odd, say* a *and* b *are odd and* c *is even.* Then ab is odd and ac and bc are even. Hence, $ab + ac + bc$ is odd.

Case 4. *All of* a, b *and* c *are odd*. Hence, ab, ac and bc are odd, as is $ab + ac + bc$.

Therefore, $ab + ac + bc$ is even if and only if at most one of a, b and c is odd. ■

7.20. In Result 6.5, it was shown that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for every positive integer n.
Evaluate the proof of the following statement.

Statement For every integer $n \geq 3$,

$$1^2 + 2^2 + \cdots + (n-1)^2 < \frac{n^3}{3} - n.$$

Proof Assume, to the contrary, that

$$1^2 + 2^2 + \cdots + (n-1)^2 \geq \frac{n^3}{3} - n$$

for every integer $n \geq 3$. By Result 6.5, it follows that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Therefore,

$$\frac{(n-1)n(2n-1)}{6} \geq \frac{n^3}{3} - n$$

and so

$$\frac{2n^3 - 3n^2 + n}{6} \geq \frac{n^3 - 3n}{3}.$$

Hence, $2n^3 - 3n^2 + n \geq 2n^3 - 6n$ and so $3n^2 - 7n = n(3n - 7) \leq 0$. Since n is a positive integer, $n \leq 7/3$, which is a contradiction. ■

7.21. Let A and B be nonempty sets. Prove that $A \times B = B \times A$ if and only if $\mathcal{P}(A) = \mathcal{P}(B)$.

7.22. The Fibonacci sequence $F_1, F_2, F_3, \ldots$ of integers is defined recursively by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for each integer $n \geq 3$. (This sequence also occurred in Exercise 6.36.) Prove that if

$a_0, a_1, a_2, \ldots$ is a sequence of rational numbers such that $a_0 = \frac{1}{2}$, $a_1 = \frac{2}{3}$ and $a_n = \frac{a_{n-2}}{a_{n-1}}$ for every integer $n \geq 2$, then for every positive integer n,

$$a_n = \begin{cases} \frac{3^{F_n}}{2^{F_{n+1}}} & \text{if } n \text{ is even} \\ \frac{2^{F_{n+1}}}{3^{F_n}} & \text{if } n \text{ is odd.} \end{cases}$$

7.23. Prove that if the real number r is a root of a polynomial with integer coefficients, then $2r$ is a root of a polynomial with integer coefficients.

7.24. Prove that if the real number r is a root of a polynomial with integer coefficients, then $r/2$ is a root of a polynomial with integer coefficients.

7.25. Prove, for every nonnegative integer n, that

$$5^{2n} + 2^{2n} \equiv 2^{2n+1} \pmod{21}.$$

7.26. Let $n \in \mathbf{Z}$. Prove that $n + 1$ and $n^2 + 3$ are of the same parity.

7.27. Prove that an integer m equals $n(n+1)/2$ for some $n \in \mathbf{N}$ if and only if $8m + 1$ is a perfect square (that is, $8m + 1 = t^2$ for some $t \in \mathbf{N}$).

7.28. By Result 4.8, if $a, b \in \mathbf{N}$ and $3 \mid ab$, then $3 \mid a$ or $3 \mid b$. Let a and n be positive integers. Prove each of the following statements.

(a) If $3 \mid a^n$, then $3 \mid a$.

(b) If $3 \mid a^n$, then $3^n \mid a^n$.

7.29. Prove that there do not exist two odd integers a and b with $a \not\equiv b \pmod 4$ such that $4 \mid (3a + 5b)$.

7.30. Prove that there exist three distinct integers $a, b, c \geq 2$ such that $a \equiv b \pmod c$, $b \equiv c \pmod a$ and $a + c \equiv 0 \pmod b$.

7.31. (a) Prove that there exists a 10-digit integer $a = a_{10}a_9 \cdots a_1$, all of whose digits are distinct, with the property that k divides $a_k a_{k-1} \cdots a_1$ for each k with $1 \leq k \leq 10$.

(b) Prove that there exists a 10-digit integer $b = b_1 b_2 \cdots b_{10}$, all of whose digits are distinct, with the property that k divides $b_1 b_2 \cdots b_k$ for each k with $1 \leq k \leq 10$.

(c) The number $n = 2468$ is a 4-digit integer with distinct digits such that the first and last digits are divisible by 1 (of course), the first and last 2-digit numbers of n, namely 24 and 68, are divisible by 2, the first and last 3-digit numbers of n are divisible by 3, and n itself is divisible by 4. Is there a 5-digit number m with the corresponding properties?

7.32. Let $S = \{1, 2, 3, 4, 5, 6\}$. Prove that there exists a collection T of five subsets of S such that for every two sets A and B in T, there is a unique set C in T for which $|A \cap C| = |B \cap C| = 1$.

7.33. According to Result 3.16, for two integers a and b, $a + b \equiv 0 \pmod 2$ if and only if $a \equiv b \pmod 2$. Let $a, b, c \in \mathbf{Z}$. Prove that $a + b + c \equiv 0 \pmod 3$ if and only if either every two integers in $\{a, b, c\}$ are congruent modulo 3 or no two integers in $\{a, b, c\}$ are congruent modulo 3.

7.34. We have seen that a triple (a, b, c) of positive integers is a Pythagorean triple if $a^2 + b^2 = c^2$. Therefore, if (a, b, c) is a Pythagorean triple, then $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$.

(a) Show that if u and v are real numbers such that $u^2 + v^2 = 1$, then $(u + v)^2 + (u - v)^2 = 2$.

(b) We saw in Result 5.30 that there are no rational solutions to the equation $x^2 + y^2 = 3$. Prove that there are infinitely many rational solutions to the equation $x^2 + y^2 = 2$.

(c) How many rational solutions to the equation $x^2 + y^2 = 4$ are there?

7.35. Prove, for every integer $n \geq 4$, that $n! > n^2$.

7.36. An office contains two tables, called Table 1 and Table 2. There are n cards on Table 1. On the bottom of each card is written a positive rational number. A total of k cards are randomly selected from Table 1 and placed on Table 2. Each of these k cards is turned over and the number on it is multiplied by $\sqrt{2}$ and turned over again. The k cards on Table 2 are then returned to Table 1 and all n cards on Table 1 are shuffled. Then once again, k cards from Table 1 are randomly selected from Table 1 and placed on Table 2. Each of these k cards is turned over and the number on it is multiplied by $\sqrt{2}$. This time, however, these k cards are left on Table 2. Suppose now that a_1 of the cards on Table 1 contain an irrational number and a_2 of the cards on Table 2 contain an irrational number. Which of the following can be said about a_1 and a_2?

(1) $a_1 < a_2$ (2) $a_1 = a_2$ (3) $a_1 > a_2$
(4) It is impossible to determine any relationship between a_1 and a_2.

7.37. Below is given a proof of a result. Which result is being proved and which proof technique is being used?

Proof Assume that $3x + 5y$ is odd. Then $3x + 5y = 2z + 1$, where $z \in \mathbf{Z}$. Then

$$\begin{aligned} 7x - 11y &= (3x + 5y) + (4x - 16y) = (2z + 1) + (4x - 16y) \\ &= 2(z + 2x - 8y) + 1. \end{aligned}$$

Since $z + 2x - 8y$ is an integer, $7x - 11y$ is odd.
For the converse, assume that $7x - 11y$ is odd. Then $7x - 11y = 2w + 1$ for some integer w. Then

$$\begin{aligned} 3x + 5y &= (7x - 11y) + (-4x + 16y) = (2w + 1) + (-4x + 16y) \\ &= 2(w - 2x + 8y) + 1. \end{aligned}$$

Since $w - 2x + 8y$ is an integer, $3x + 5y$ is odd. ■

7.38. Below is given a proof of a result. Which result is being proved and which proof technique is being used?

Proof Assume, to the contrary, that there are integers a and b such that $a^2 - 4b^2 = 2$. Certainly, a is not odd, for otherwise, a^2 and $a^2 - 4b^2$ are odd and so $a^2 - 4b^2 \neq 2$. Thus, a must be even and so $a = 2c$ for some integer c. Therefore,

$$a^2 - 4b^2 = (2c)^2 - 4b^2 = 4c^2 - 4b^2 = 4(c^2 - b^2) = 2.$$

Since $c^2 - b^2$ is an integer, $4 \mid 2$, which is impossible. ■

7.39. Prove that there exist three distinct real number solutions to the polynomial equation $x^3 - 3x + 1 = 0$.

7.40. Prove that there exists no integer a for which $a \equiv 17 \pmod{35}$ and $2a \equiv 43 \pmod{49}$.

7.41. We have seen that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for every positive integer n. The statement below suggests that there is another expression for this sum. Evaluate the proof of the following statement.

Statement For every positive integer n, $\sum_{i=1}^{n} i = \frac{(2n+1)^2}{8}$.

Proof We proceed by induction. First, observe that the statement is true for $n = 1$. Assume that the statement is true for a positive integer k. Thus, $\sum_{i=1}^{k} i = \frac{(2k+1)^2}{8}$. We show that $\sum_{i=1}^{k+1} i = \frac{(2k+3)^2}{8}$. Now,

$$\begin{aligned} \sum_{i=1}^{k+1} i &= \left(\sum_{i=1}^{k} i\right) + (k+1) = \frac{(2k+1)^2}{8} + (k+1) \\ &= \frac{4k^2 + 4k + 1 + 8(k+1)}{8} = \frac{4k^2 + 12k + 9}{8} = \frac{(2k+3)^2}{8}. \end{aligned}$$

Thus, $\sum_{i=1}^{n} i = \frac{(2n+1)^2}{8}$ for every positive integer n by the Principle of Mathematical Induction. ■

7.42. Evaluate the proof of the following statement.

Statement For each positive integer n, every set of n real numbers consists only of equal numbers.

Proof We proceed by induction. Certainly, if a set consists of a single real number, then all numbers in the set are equal. Assume, for a positive integer k, that the numbers in every set of k real numbers are equal. Let S be a set of $k+1$ real numbers, say $S = \{a_1, a_2, \dots, a_{k+1}\}$. Let $S_1 = \{a_1, a_2, \dots, a_k\}$ and $S_2 = \{a_2, a_3, \dots, a_{k+1}\}$ be two subsets of S, each consisting of k real numbers. By the induction hypothesis, all numbers in S_1 are equal and numbers in S_2 are equal, that is, $a_1 = a_2 = \cdots = a_k$ and $a_2 = a_3 = \cdots = a_{k+1}$. Therefore, $a_1 = a_2 = \cdots = a_k = a_{k+1}$ and so all numbers in S are equal. By the Principle of Mathematical Induction, every set of n real numbers consists only of equal numbers for every positive integer n. ■

7.43. Evaluate the proof of the following statement.

Statement For every nonnegative integer n, $e^n = 1$.

Proof We proceed by the Strong Principle of Mathematical Induction. First, since $e^0 = 1$, the statement is true for $n = 0$. Assume, for a nonnegative integer k, that $e^i = 1$ for every integer i with $0 \le i \le k$. We show that $e^{k+1} = 1$. Observe that

$$e^{k+1} = \frac{e^k \cdot e^k}{e^{k-1}} = \frac{1 \cdot 1}{1} = 1.$$

By the Strong Principle of Mathematical Induction, $e^n = 1$ for every nonnegative integer n. ■

7.44. Prove that if a is a real number such that $|a| < r$ for every positive real number r, then $a = 0$.

7.45. Below is given a proof of a result. Which result is being proved and which proof technique is being used?

Proof First, observe that $3^0 + 1 = 2 \ge 1^2 = 1$, which verifies the basis step. Assume that $3^k + 1 \ge (k+1)^2$ for some nonnegative integer k. We show that $3^{k+1} + 1 \ge (k+2)^2$. When $k = 0$, we have $3^1 + 1 = 4 = 2^2$. Hence, we may assume that k is a positive integer. Observe that

$$\begin{aligned} 3^{k+1} + 1 &= 3 \cdot 3^k + 1 \ge 3[(k+1)^2 - 1] + 1 \\ &= 3(k^2 + 2k) + 1 = 3k^2 + 6k + 1 = (k^2 + 4k) + (2k^2 + 2k) + 1 \\ &\ge k^2 + 4k + 2 + 2 + 1 \ge k^2 + 4k + 4 = (k+2)^2. \end{aligned}$$ ■

7.46. Evaluate the proof of the following statement.

Statement If $a_1, a_2, \dots, a_n$ are n real numbers such that $a_1 a_2 \cdots a_n = 0$, then $a_i = 0$ for some i with $1 \le i \le n$.

Proof We proceed by induction. Certainly, the statement is true for $n = 1$. Assume that the statement is true for some positive integer k. Now, let $b_1, b_2, \dots, b_{k+1}$ be $k+1$ real numbers such that $b_1 b_2 \cdots b_{k+1} = 0$. Thus, $(b_1 b_2 \cdots b_k) b_{k+1} = 0$ and hence either $b_1 b_2 \cdots b_k = 0$ or $b_{k+1} = 0$. If $b_1 b_2 \cdots b_k = 0$, then it follows by the induction hypothesis that $b_i = 0$ for some integer i with $1 \le i \le k$. If this is not the case, then $b_{k+1} = 0$. Hence, $b_i = 0$ for some integer i with $1 \le i \le k+1$. Therefore, the statement is true by the Principle of Mathematical Induction. ■

7.47. Evaluate the proof of the following statement.

Statement If $n \ge 10$ is an integer, then $n^3 \ge 100 + 9n^2$.

Proof First, observe that if $n = 10$, then $n^3 = 1000$ and $100 + 9n^2 = 100 + 900 = 1000$ and so $n^3 = 100 + 9n^2$. More generally, observe that $n^3 \ge 100 + 9n^2$ can be written as $n^3 - 9n^2 \ge 100$ and so $n^2(n-9) \ge 100$. Since $n \ge 10$, we have $n^2 \ge 100$ and $n - 9 \ge 1$. Therefore, $n^2(n-9) \ge 100 \cdot 1 = 100$. ■