Exercises

1. Let $f, g, h : X \to \mathbb{R}$ be functions such that, for every $x \in X$ we have $f(x) \leq g(x) \leq h(x)$. Show that if there is a point $a \in X \cap X'$ such that $f(a) = h(a)$ and $f'(a) = h'(a)$ then $g'(a)$ exists and $g'(a) = f'(a) = h'(a)$.

2. Let $p : \mathbb{R} \to \mathbb{R}$ be an odd degree polynomial. Then there exists $c \in \mathbb{R}$ such that $p''(c) = 0$.

3. Let $f : X \to \mathbb{R}$ be differentiable at $a \in X \cap X'$. If x_n and y_n are sequences in X such that $\lim x_n = \lim y_n = a$ and $x_n < a < y_n$ for every $n \in \mathbb{N}$, show that

$$\lim \frac{f(y_n) - f(x_n)}{y_n - x_n} = f'(a).$$

4. Show that the function given by $f(0) = 0$, $f(x) = x^2 \sin \frac{1}{x}$ if $x \neq 0$, is differentiable. Find sequences x_n and y_n such that $x_n \neq y_n$, $\lim x_n = \lim y_n = 0$ but $\lim \frac{f(y_n)-f(x_n)}{y_n-x_n}$ doesn't exist.

5. Let $f : I \to \mathbb{R}$ be differentiable on an interval $I \subseteq \mathbb{R}$. We call $a \in I$ a *critical point* if $f'(a) = 0$. We say a critical point a is *non-degenerate* if $f''(a) \neq 0$.

   5.1 If $f \in C^1$, show that the set of all critical points contained in a closed interval $[c, d] \subseteq I$ is closed.

   5.2 Show that local maximum and minimum points of f are critical points. Moreover, any critical non-degenerate point is a maximum or minimum.

   5.3 Show that there are C^∞ functions with isolated degenerate local maximum/minimums. Moreover, there are critical points of C^∞ functions that are not local maximum/minimum points.

   5.4 Show that every non-degenerate critical point of f is isolated.

   5.5 Let $f \in C^1$, suppose that the critical points of f contained in a closed interval $[c, d] \subseteq I$ are non-degenerate. Show that there are finitely many of them. Conclude that f has at most a countable number of non-degenerate critical points in I.

   5.6 The function $f(0) = 0$, $f(x) = x^4 \sin \frac{1}{x}$ if $x \neq 0$ has infinitely many non-degenerate critical points in $[0, 1]$. Wouldn't this be a contradiction to 5.4? Why/why not?

6. Let $f : I \to \mathbb{R}$ be a function defined on interval $I \subseteq \mathbb{R}$. If there is $C, \alpha > 0$ such that $\forall x, y \in I \Rightarrow |f(x) - f(y)| \leq C|x - y|^\alpha$, we say f is *Holder continuous*. Show that if $\alpha > 1$ then f is constant.

7. Let $f : I \to \mathbb{R}$ be differentiable on an interval $I \subseteq \mathbb{R}$. Show that if $f'(x) = 0$ for every $x \in I$ then f is constant.

8. Show that a differentiable function $f : I \to \mathbb{R}$ is Lipschitz, i.e. $|f(x) - f(y)| \leq C|x - y|$, if and only if $|f'(x)| \leq C$.

9. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $f \in C^\infty$, $f(x) \neq x$, $\forall x \in \mathbb{R}$ and $|f'(x)| < 1$.

10. Let $f : [0, \pi] \to \mathbb{R}$ be defined by $f(x) = \cos(\cos(x))$. Show that $|f'(x)| \leq c < 1$ for some $c \in \mathbb{R}$.

11. Let $f : (a, +\infty) \to \mathbb{R}$ be differentiable. Show that if $\lim_{x \to +\infty} f(x) = b$ and $\lim_{x \to +\infty} f'(x) = c$, then $c = 0$. *[Hint: Apply the Mean Value theorem on $[n, n+1]$ and let $n \to +\infty$.]*

12. Let $f : [a, b] \to \mathbb{R}$ be continuous, differentiable on (a, b), satisfying $f(a) = f(b)$. Given $k \in \mathbb{R}$, show that $\exists c \in (a, b)$ such that $f'(c) = kf(c)$. *[Hint: Apply Rolle's theorem to $g(x) = f(x)e^{-kx}$.]*

13. Let $f : I \to \mathbb{R}$ be differentiable on an interval $I \subseteq \mathbb{R}$. A *root* of f is a number $c \in I$ such that $f(c) = 0$. Show that between two consecutives roots of f', there is at most one root of f.

14. Let $f : [0, +\infty) \to \mathbb{R}$ be twice differentiable. Show that if f'' is bounded and $\lim_{x \to +\infty} f(x)$ exists, then $\lim_{x \to +\infty} f'(x) = 0$.

15. Show that the composition of C^k functions is still a C^k function.

16. Given $a, b \in \mathbb{R}$ with $a < b$, consider $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(x) = \begin{cases} e^{\frac{1}{(x-a)(x-b)}}, & \text{if } x \in (a, b), \\ 0, & \text{if } x \notin (a, b). \end{cases}$$

Show that $\varphi \in C^\infty$ and φ has exactly one maximum point.

17. Let $f : I \to \mathbb{R}$ be twice differentiable at $a \in I^\circ$. Show that

$$f''(a) = \lim_{h \to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2}$$

Given a example where the limit above exists but $f'(a)$ doesn't.

18. Show that the function $f(x) = |x|^{2n+1}$ is of class C^{2n} but $f^{(2n+1)}(x)$ doesn't exist in every $a \in \mathbb{R}$.