Exercises

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that the *zero set* of f

$$Z(f) = \{x; f(x) = 0\}$$

is a closed set. Conclude that if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous then the zero set $\{x; f(x) = g(x)\}$ is closed.

2. Let $f : X \to \mathbb{R}$ be continuous. Show that for every $k \in \mathbb{R}$, the set of all $x \in X$ such that $f(x) \leq k$ is of the form $C \cap X$, where C is closed.

3. Let $f : X \to \mathbb{R}$ be a function and $X \subseteq \mathbb{R}$ an open set. Show that f is continuous if and only if the sets $\{x; f(x) < c\}$ and $\{x; f(x) > c\}$ are open for every $c \in \mathbb{R}$.

4. Let $f : X \to \mathbb{R}$ be a function and $X \subseteq \mathbb{R}$ an open set. Show that f is continuous if and only if the set $f^{-1}(A)$ is open for every open $A \subseteq \mathbb{R}$.

5. Let $f : X \to \mathbb{R}$ be a function and $X \subseteq \mathbb{R}$ a closed set. Show that f is continuous if and only if the set $f^{-1}(C)$ is closed for every closed set $C \subseteq \mathbb{R}$.

6. Let $S \subseteq \mathbb{R}$ be nonempty. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \inf\{|x - s|; s \in S\}$$

Show that f is *Lipschitz*: $\forall x, y \in \mathbb{R} \Rightarrow |f(x) - f(y)| \leq |x - y|$.

7. Let $X \subseteq \mathbb{R}$ be a closed set and $f : X \to \mathbb{R}$ continuous. Show that there exist a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g_{|X} = f$.

8. Give an example of a bijective function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous at every $a \in \mathbb{R}$.

9. Show that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ that takes every rational number to an irrational number, and vice-versa.

Solution. Suppose there is such a function. We know that if f is continuous, it takes intervals to intervals. In particular, $f(\mathbb{R})$ should be an interval, since $\mathbb{R} = (-\infty, +\infty)$. However, $f(\mathbb{R}) = f(\mathbb{Q} \cup \mathbb{Q}^c) = f(\mathbb{Q}) \cup f(\mathbb{Q}^c) \subseteq f(\mathbb{Q}) \cup \mathbb{Q}$, a contradiction since $f(\mathbb{Q}) \cup \mathbb{Q}$ is countable, hence can't contain an interval (which is uncountable). $\square$

10. Let A be the set of all nonnegative algebraic numbers, and B be the set of negative transcendental numbers. Let $f : A \cup B \to [0, +\infty)$ be a function defined by $f(x) = x^2$. Show that f is a continuous bijection, whose inverse f^{-1} is discontinuous at every point, except zero.

11. (Brouwer Fixed Point Theorem) Let $f : [a, b] \to [a, b]$ be a continuous function. Show that there exists a point $x \in [a, b]$ such that $f(x) = x$. *[We call such point a 'fixed point'.]*

Solution. Set $g(x) = f(x) - x$. Then since $f(a) \geq a$, we have $g(a) \geq 0$. Similarly, $f(b) \leq b \Rightarrow g(b) \leq 0$. The result follows from the intermediate value theorem. $\square$

12. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If for every open set $A \subseteq \mathbb{R}$, the image $f(A)$ is open, then f is injective, hence monotone.

13. Fix $X \subseteq \mathbb{R}$. If every function defined on X is bounded then X is compact.

14. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = +\infty$. Then f achieves its minimum value, i.e. there is $a \in \mathbb{R}$ such that $f(a) \leq f(x), \forall x \in \mathbb{R}$.

15. Show that $f : (-1, 1) \to \mathbb{R}$ given by $f(x) = \frac{1}{1-|x|}$ is a homeomorphism.

16. Classify all intervals of $\mathbb{R}$ up to homeomorphism. For example, all open intervals, whether or not bounded, are homeomorphic, hence should represent the same object.

    Solution. Let X be the set of all intervals I in $\mathbb{R}$ up to homeomorphism, we claim

    $$X = \{(0, 1), [0, 1], [0, 1)\}.$$

    Indeed, if I is open then it is homeomorphic to $(0, 1)$. If I is closed and bounded then it is homeomorphic to $[0, 1]$. If I is closed and unbounded or I is half-open then it is homeomorphic to $[0, 1)$. $\square$

17. Show that the inverse of f given in exercise 15, is uniformly continuous. (Notice that f isn't)

18. Show that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin x$ is uniformly continuous, but $g(x) = \sin x^2$ isn't.

19. Show that a polynomial $p : \mathbb{R} \to \mathbb{R}$ is uniformly continuous if and only if has degree at most one.

20. Show that $f(x) = x^n$ is Lipschitz in any bounded set. Moreover, prove that if $n > 1$ and f is defined on an unbounded interval, then f is not even uniformly continuous.

21. Give an example of sets A, B open and a continuous function $f : A \cup B \to \mathbb{R}$ such that $f_{|A}, f_{|B}$ are uniformly continuous but f is not.

22. Given a function $f : X \to \mathbb{R}$. Suppose that for every $\epsilon > 0$, there exists $g : X \to \mathbb{R}$ continuous, such that $\forall x \in X$, $|f(x) - g(x)| < \epsilon$. Show that f is continuous.