Quasilinear Hardy-Henón equation with power source

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Abstract

We discuss the existence (or non existence) of **nontrivial** solutions to a generalized Hardy-Henón type equation given by

$$\begin{cases} -\Delta_p u = |x|^\sigma |u|^{q-2}u + \lambda u^r & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $1 < p < N, 2 \leq q < p^*, 1 < r + 1 < p^*, \max\{-N, -q\} < \sigma$ and $\lambda \in \mathbb{R}$ is a fixed constant.

Keywords: Hardy-Henon's equation, quasilinear, p-laplacian

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1 Introduction

The equation we study in this manuscript is a generalization of the usual Hénon equation

$$-\Delta u = |x|^\sigma u^q, \; x \in \mathbb{R}^n, \sigma \in \mathbb{R}. \tag{2}$$

Introduced in [12], the equation above models rotating stellar systems. Despite being first published in a physics journal, the mathematical community was quickly attracted to this equation due its many interesting properties, like singularity if $\sigma < 0$ and unboundedness if $\sigma > 0$, in which case the equation is also known as Hardy equation. The first rigorous mathematical analysis of this equation was done in [14].

When $\sigma = 0$, this equation is known as Lane-Emden equation. In this case, the equation becomes somewhat easier to study and existence and regularity of solutions

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were known even before the Hénon equation. The Lane-Emden equation is one of the simplest semilinear elliptic equations, and has been extensively studied over the years both in bounded and unbounded domains.

During the past decades, many generalizations of this equation were proposed and analyzed. For example, in [10, 2] the authors studied the quasi-linear generalization of (2) using the p-laplacian. In [1, 11] the authors study a type of parabolic counter-part of equation (2). A summary of what is known in the semilinear unbounded domain case $\Omega = \mathbb{R}^n$ has been compiled in [9].

In this work we plan to study existence (or nonexistence) of solutions to a generalization of equation (2) in a bounded domain $\Omega \subseteq \mathbb{R}^N$.

Henceforward, we will focus on the following Dirichlet problem

$$\begin{cases} -\Delta_p u = |x|^\sigma |u|^{q-2}u + \lambda u^r & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{*}$$

where $1 < p < N, 2 \leq q < p^*, 1 < r+1 < p^*, \max\{-N, -q\} < \sigma$ and $\lambda \in \mathbb{R}$ is a fixed constant. Here,$-\Delta_p u := \operatorname{div}(|Du|^{p-2}Du)$ denotes the p-laplacian operator. Notice that if $p = q = 2$ and $\lambda = 0$ we recover the classical Hénon equation (2).

The condition $\max\{-N, -q\} < \sigma$ seems random and unmotivated, however it has a simple explanation, both conditions are required by the Caffarelli-Kohn-Niremberg inequality described below. In particular, $-N < \sigma$ is a requirement for the integrability of the function $|x|^\sigma$ on a bounded domain containing the origin.

We define a **weak solution** to equation (*) as a function $u \in \mathbf{W}_0^{1,p}(\Omega)$ such that

$$\int_\Omega |Du|^{p-2} Du D\varphi = \int_\Omega |x|^\sigma |u|^{q-2} u\varphi + \lambda \int_\Omega u^r \varphi \quad \forall \varphi \in \mathbf{W}_0^{1,p}(\Omega) \tag{3}$$

Weak solutions can be described as critical points of the functional

$$I(u) = \frac{1}{p}\int_\Omega |Du|^p - \frac{1}{q}\int_\Omega |x|^\sigma |u|^q - \frac{\lambda}{r+1}\int_\Omega u^{r+1} \tag{4}$$

Notice that if $2 \leq q < p$ and $1 < r+1 < p$, the functional above satisfies

$$I(u) \to +\infty \text{ when } \|u\| \to +\infty$$

and hence is coercive, whereas when $p < q < p^*$ and $p < r+1 < p^*$, $I(u)$ is not coercive.

Even if $I(u)$ is coercive is not necessarily strictly convex, hence the direct method could show the existence of the trivial solution, which we already know exists by inspection. So a different approach is needed to prove the existence of nontrivial solutions.

For the convenience of the reader, we summarize the main results in this paper.

In section 2.1, the case $\lambda < 0$ is analyzed and we are able to prove existence of solutions that are actually extremal function of the CKN inequality (5). A Non-existence result is also proved in 2.1. In section 2.2, we study the case $\lambda > 0$ and prove existence results. For the sake of completeness, we discuss the simple case when $\lambda = 0$ in 2.3 and prove existence of a nontrivial solution in this case as well. Lastly, in 3 we discuss some open questions related to this paper.

Notation & Assumptions

- $\Omega \subseteq \mathbb{R}^N$ is a bounded domain and $N \geq 3$.
- The space $\mathbf{W}_0^{1,p}(\Omega)$ denotes the usual Sobolev space which is the closure of $\mathcal{C}_0^\infty(\Omega)$, smooth functions with compact support using the Sobolev norm.
- For $q > 1$, q' denotes the Holder conjugate, i.e. $\frac{1}{q} + \frac{1}{q'} = 1$, and q^* denotes the Sobolev conjugate, defined by $q^* = \frac{qN}{N-q} > q$.
- The letter C will always denote a positive constant which may vary from place to place.
- The Lebesgue measure of a set $A \subseteq \mathbb{R}^n$ is denoted by $|A|$.
- The symbol $\rightharpoonup$ denotes weak convergence.

2 Proofs of the main results

In this manuscript, we make frequent use of the following short version of the Caffarelli-Kohn-Nirenberg inequality, which has been adapted to our purposes.
Theorem 1. *(Caffarelli-Kohn-Nirenberg [3]) There exists a positive constant C such that for every $u \in C_0^\infty(\mathbb{R}^n)$:*

$$\left(\int |x|^\sigma |u|^q\right)^{\frac{1}{q}} \leq C \left(\int |Du|^p\right)^{\frac{a}{p}} \left(\int |u|^{r+1}\right)^{\frac{1-a}{r+1}} \tag{5}$$

where $-N < \sigma \leq 0, q > 0, p > 1, r > 0, 0 < a \leq 1$ and

$$a\left(\frac{1}{p} - \frac{1}{N}\right) + \frac{1-a}{r+1} = \frac{1}{q} + \frac{\sigma}{qN} \tag{6}$$

If $a = 1$, we must have $\sigma \geq -q$ as well.

2.1 The case $\boldsymbol{\lambda < 0}$

In the linear case, $p = 2$, a similar problem (in an unbounded domain) with $r < 1$ and $q = 2$ has been recently analyzed in [13]. The results in this section provide a generalization of the latter in a bounded domain setting.

Let $\lambda = -\mu$, with $\mu > 0$. The goal of this section is to prove the existence of a nontrivial weak solution ($u \neq 0$) to

$$\begin{cases} -\Delta_p u + \mu u^r = |x|^\sigma |u|^{q-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

We use constrained minimization to find a nontrivial solution. Namely, consider the set

$$S := \left\{ u \in \mathbf{W}_0^{1,p}(\Omega); \int_\Omega |x|^\sigma |u|^q = 1 \right\}, \tag{8}$$

and define the operator

$$J(u) = \frac{1}{p} \int_\Omega |Du|^p + \frac{\mu}{r+1} \int_\Omega |u|^{r+1}$$

We first prove that the infimum

$$m = \inf_{u \in S} J(u)$$

is achieved by a function $v \in S$.

Lemma 2. *There is a nonnegative nontrivial function $v \in S$ such that $m = J(v)$.*

Proof. Let $u_n \in S$ be a minimizing sequence. Since $|u_n|$ is also a minimizing sequence, we may assume without loss of generality that $u_n \geq 0$.

Over the set S, $J(u)$ is coercive, hence u_n is bounded. We may assume that, taking a subsequence if necessary, $u_n \rightharpoonup v$ for some $v \in \mathbf{W}_0^{1,p}(\Omega)$. Using the weak lower semi-continuity of the norm, we have:

$$J(v) \leq \liminf J(u_n) = m$$

We claim next that $\int_\Omega |x|^\sigma |v|^q = 1$. If $\sigma \geq 0$ we can easily pass the limit on

$$\int_\Omega |x|^\sigma |u_n|^q = 1$$

using Lebesgue's dominated convergence theorem and the fact that $q < p^*$.

Suppose $\sigma < 0$. Notice that by (5), we can find $a \in (0,1)$ such that:

$$\left(\int_\Omega |x|^\sigma |u_n - v|^q \right)^{\frac{1}{q}} \leq C \left(\int_\Omega |D(u_n - v)|^p \right)^{\frac{a}{p}} \left(\int_\Omega |u_n - v|^{r+1} \right)^{\frac{1-a}{r+1}}$$

Using the fact that $r + 1 < p^*$ and boundedness of $\int_\Omega |D(u_n - v)|^p$, we deduce that

$$\lim \left(\int_\Omega |x|^\sigma |u_n - v|^q \right)^{\frac{1}{q}} = 0$$

Therefore, $\int_\Omega |x|^\sigma |v|^q = 1$. We conclude that $v \in S$ and $J(v) = m$. $\square$

Theorem 3. *Suppose $1 < p < N, 2 \leq q < p^*, q \neq r + 1 < p^*$ and $\max\{-N, -q\} < \sigma$, then there is a nonnegative nontrivial solution to problem (7).*

Proof. We claim that the function $v \in \mathbf{W}_0^{1,p}(\Omega)$ obtained in lemma 2, up to a transformation, is the desired solution. Indeed, for $s \in \mathbb{R}$ sufficiently small we can assume $v + su$ is not identically zero for a fixed $u \in \mathbf{W}_0^{1,p}(\Omega)$. Then, we can find $t(s)$ such that

$$\int_\Omega |x|^\sigma |t(s)(v+su)|^q = 1.$$

Namely,

$$t(s) = \left(\int_\Omega |x|^\sigma |(v+su)|^q \right)^{-\frac{1}{q}}.$$

Hence, the application $s \mapsto t(s)(v+su)$ defines a curve on S with the property that $0 \mapsto v$. If we set $\gamma(s) = J(t(s)(v+su))$, it follows that $s = 0$ is a local minimum for $\gamma(s)$, hence

$$0 = \gamma'(0) = J'(v)[t'(0)v + u]$$

Simplifying, we obtain

$$J'(v)u = -t'(0)J'(v)v.$$

Since

$$-t'(0) = \int_\Omega |x|^\sigma |v|^{q-2} vu,$$

we conclude that (recall that $v \geq 0$):

$$\int_\Omega |Dv|^{p-2} DvDu + \mu \int_\Omega v^r u = J'(v)v \int_\Omega |x|^\sigma |v|^{q-2} vu.$$

Set $\nu := J'(v)v > 0$ and consider the function $w(x) = \nu^i v(\nu^j x)$, for some $i, j \in \mathbb{R}$ to be determined. Then:

$$\nu^{-(i+j)-(p-2)(i+j)} \int_\Omega |Dw|^{p-2} DwDu + \mu \nu^{-ri} \int_\Omega w^r u = \nu^{1-i-i(q-2)} \int_\Omega |x|^\sigma |w|^{q-2} wu.$$

It follows that if $r \neq q-1$, we may set $i = -\frac{1}{r+1-q}$ and $j = -\frac{r+1-p}{(p-1)(r+1-q)}$ so $w = \nu^i v(\nu^j x)$ is the desired nonnegative nontrivial solution to equation (7). □

Corollary 4. *If the conditions of theorem 3 hold and moreover, $\sigma < 0$ and $r + 1 < q$, then the solution $v(x)$ described above is also a extremal function of the CKN inequality (5), i.e. v achieves the infimum*

$$\inf_{\substack{u \in \mathbf{W}_0^{1,p}(\Omega) \\ u \neq 0}} \frac{\|Du\|_p^a \|u\|_{r+1}^{1-a}}{\left(\int |x|^\sigma |u|^q \right)^{\frac{1}{q}}}$$

where the parameter ' a ' satisfies (6).

Proof. Given $n > 0$ and $u \in \mathbf{W}_0^{1,p}(\Omega)$ not identically zero, set

$$u_n(x) = nu\left(n^{\frac{q}{N+\sigma}}\left(\int_\Omega |x|^\sigma |u|^q\right)^{\frac{1}{N+\sigma}} x\right)$$

It's immediate to check that

$$\int_\Omega |x|^\sigma |u_n|^q = 1,$$

hence $u_n \in S$ (recall that S is defined by (8)). From now on we set

$$N(u) := \int_\Omega |x|^\sigma |u|^q.$$

We already know that $J(v) \leq J(u_n)$ for every $n > 0$. It follows that:

$$N(u)^{\frac{N}{N+\sigma}} \leq \frac{J(u_n)}{J(v)} N(u)^{\frac{N}{N+\sigma}} =: P(n,u) \tag{9}$$

Simplifying we have:

$$\begin{aligned} P(n,u) &= \left(\frac{\|Du\|_p^p}{pJ(v)}\right) n^{\frac{p(N+\sigma)+q(p-N)}{N+\sigma}} + \left(\frac{\mu\|u\|_{r+1}^{r+1}}{(r+1)J(v)}\right) n^{\frac{(r+1)(N+\sigma)-qN}{N+\sigma}} \\ &= A(u)n^\alpha + B(u)n^{-\beta} \end{aligned}$$

where

$$\begin{aligned} A(u) &:= \frac{N(u)^{\frac{p}{N+\sigma}} \|Du\|_p^p}{pJ(v)} \text{ and } B(u) := \frac{\mu\|u\|_{r+1}^{r+1}}{(r+1)J(v)}, \\ \alpha &= \frac{p(N+\sigma)+q(p-N)}{N+\sigma} > 0, \\ \beta &= \frac{qN-(r+1)(N+\sigma)}{N+\sigma} > 0. \end{aligned}$$

After a quick computation, we observe that

$$n^* := \left[\frac{\beta B(u)}{\alpha A(u)}\right]^{\frac{1}{\alpha+\beta}} \text{ satisfies } P(n^*,u) = \inf_{n>0} P(n,u) \tag{10}$$

If we set $f(u) = P(n^*,u)$, then since $A(v) + B(v) = 1$ we obtain f(v)=1. Also,

$$f(u) = P(n^*,u) = (\alpha+\beta)\alpha^{\frac{-\alpha}{\alpha+\beta}} \beta^{\frac{-\beta}{\alpha+\beta}} A(u)^{\frac{\beta}{\alpha+\beta}} B(u)^{\frac{\alpha}{\alpha+\beta}} \tag{11}$$

Therefore, from (9),(10),(11) we obtain:

$$\begin{aligned} N(u)^{\frac{N(\alpha+\beta)}{N+\sigma}} &\leq \frac{(\alpha+\beta)^{\alpha+\beta}}{\alpha^\alpha\beta^\beta} A(u)^\beta B(u)^\alpha \\ &\leq \frac{(\alpha+\beta)^{\alpha+\beta}}{\alpha^\alpha\beta^\beta} \frac{N(u)^{\frac{p\beta}{N+\sigma}}\|Du\|_p^{p\beta}}{p^\beta J(v)^\beta} \frac{\mu^\alpha\|u\|_{r+1}^{\alpha(r+1)}}{(r+1)^\alpha J(v)^\alpha} \\ &= KN(u)^{\frac{p\beta}{N+\sigma}}\|Du\|_p^{p\beta}\|u\|_{r+1}^{\alpha(r+1)} \end{aligned}$$

where

$$K = \frac{(\alpha+\beta)^{\alpha+\beta}}{\alpha^\alpha\beta^\beta J(v)^{\alpha+\beta} p^\beta (r+1)^\alpha}.$$

It follows that:

$$N(u)^{\frac{1}{q}} \leq K^{\frac{N+\sigma}{q[N(\alpha+\beta)-p\beta]}}\|Du\|_p^{\frac{p\beta(N+\sigma)}{q[N(\alpha+\beta)-p\beta]}}\|u\|_{r+1}^{\frac{\alpha(r+1)(N+\sigma)}{q[N(\alpha+\beta)-p\beta]}}.$$

Interestingly enough, if we set

$$a := \frac{p\beta(N+\sigma)}{q[N(\alpha+\beta)-p\beta]},$$

then $\frac{\alpha(r+1)(N+\sigma)}{q[N(\alpha+\beta)-p\beta]} = 1-a$, and we conclude that

$$K^{-\frac{N+\sigma}{q[N(\alpha+\beta)-p\beta]}} \leq \frac{\|Du\|_p^a\|u\|_{r+1}^{1-a}}{N(u)^{\frac{1}{q}}}$$

with equality if $u = v$. The conclusion follows. $\square$

Theorem 5. *Suppose* $1 < p < N, p \leq q \leq p^*, r+1 \leq q$ *and* $\sigma \leq -q$, *then there is no nontrivial nonnegative solution* $u(x) \in \mathbf{W}_0^{1,p}(\Omega)$ *to problem* (7) *with the property that* $u \in W^{2,p}_{loc}(\Omega)$.

Proof. The idea behind the proof is to integrate by parts *à la Pohozaev*, namely, we multiply (7) by $x \cdot Du$ and integrate. We obtain

$$\int_\Omega |Du|^{p-2}DuD(x\cdot Du) + \mu\int_\Omega u^r(x\cdot Du) = \int_\Omega |x|^\sigma|u|^{q-2}u(x\cdot Du)$$

We analyze each integral above separately, first we have

$$\begin{aligned}
\int_\Omega |Du|^{p-2} Du D(x\cdot Du) &= \int_\Omega |Du|^{p-2} Du (x\otimes D^2u + Du\otimes I_n) \\
&= \int_\Omega |Du|^{p-2} Du (x\otimes D^2u) + \int_\Omega |Du|^{p-2}|Du|^2 \\
&= \frac{1}{p}\int_\Omega x\cdot D|Du|^p + \|Du\|_p^p \\
&= -\frac{N}{p}\int_\Omega |Du|^p + \|Du\|_p^p \\
&= \frac{p-N}{p}\|Du\|_p^p
\end{aligned} \tag{12}$$

Next, we have:

$$\begin{aligned}
\mu\int_\Omega u^r(x\cdot Du) &= \frac{\mu}{r+1}\int_\Omega x\cdot Du^{r+1} \\
&= -\frac{N\mu}{r+1}\int_\Omega u^{r+1} \\
&= -\frac{N\mu}{r+1}\|u\|_{r+1}^{r+1}
\end{aligned} \tag{13}$$

Finally,

$$\begin{aligned}
\int_\Omega |x|^\sigma |u|^{q-2}u(x\cdot Du) &= \frac{1}{q}\int_\Omega |x|^\sigma x\cdot D|u|^q \\
&= -\frac{N+\sigma}{q}N(u) \\
&= -\frac{N+\sigma}{q}\left(\|Du\|_p^p + \mu\|u\|_{r+1}^{r+1}\right)
\end{aligned} \tag{14}$$

Combining (12),(13),(14) we obtain:

$$\frac{p-N}{p}\|Du\|_p^p - \frac{N\mu}{r+1}\|u\|_{r+1}^{r+1} = -\frac{N+\sigma}{q}\left(\|Du\|_p^p + \mu\|u\|_{r+1}^{r+1}\right)$$

We conclude that

$$\left(\frac{p-N}{p} + \frac{N+\sigma}{q}\right)\|Du\|_p^p + \left(\frac{\mu(N+\sigma)}{q} - \frac{N\mu}{r+1}\right)\|u\|_{r+1}^{r+1}$$

It follows that if $\sigma \le -q, p\le q$ and $r+1\le q$, we must have $u\equiv 0$. $\square$

Corollary 6. *Suppose $1<p<N, p\le q\le p^*, r+1\le q$ and $\sigma\le -q$, then there is no nontrivial nonnegative classical solution $u(x)\in C^2(\Omega)$ to problem (7).*

2.2 The case $\lambda > 0$

Recall, that the problem we are analyzing is

$$\begin{cases} -\Delta_p u = |x|^\sigma |u|^{q-2}u + \lambda u^r & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

where we assume henceforward that $\lambda > 0$. It follows from the strong maximum principle that any nontrivial nonnegative solution is actually positive in Ω.

Our goal in this section is to prove existence of positive solutions.

When $p > r+1$, the operator

$$J(u) = \frac{1}{p}\int_\Omega |Du|^p - \frac{\lambda}{r+1}\int_\Omega |u|^{r+1}$$

is still coercive when we take u in the set

$$S := \left\{ u \in \mathbf{W}_0^{1,p}(\Omega); \int_\Omega |x|^\sigma |u|^q = 1 \right\},$$

Therefore, the method develop in subsection 2.1 can be applied *mutadis mutandis* and we easily obtain

Theorem 7. *Suppose* $1 < p < N, 2 \le q < p^*, q \neq r+1 < p$ *and* $\max\{-N, -q\} < \sigma$, *then there is a positive solution to problem* (15).

Now, let's suppose $p \le r+1 < p^*$, and consider the operator

$$I(u) = \frac{1}{p}\int_\Omega |Du|^p - \frac{1}{q}\int_\Omega |x|^\sigma |u|^q - \frac{\lambda}{r+1}\int_\Omega u^{r+1}, \tag{16}$$

which is defined and differentiable on $\mathbf{W}_0^{1,p}(\Omega)$.

Consider the Nehari Manifold associated to $I(u)$, namely:

$$\begin{aligned} \mathcal{N} &:= \{u \in \mathbf{W}_0^{1,p}(\Omega); u \not\equiv 0 \text{ and } I'(u)u = 0\} \\ &= \{u \in \mathbf{W}_0^{1,p}(\Omega); u \not\equiv 0 \text{ and } \|Du\|_p^p = \int_\Omega |u|^\sigma |u|^q + \lambda \int_\Omega u^{r+1}\} \end{aligned}$$

If we assume for the moment that this set is nonempty, the functional $I(u)$ restricted to $\mathcal{N}$ becomes

$$\begin{aligned} I(u) &= \left(\frac{1}{p} - \frac{1}{q}\right)\int_\Omega |Du|^p + \lambda\left(\frac{1}{q} - \frac{1}{r+1}\right)\int_\Omega u^{r+1}, \\ &= \left(\frac{1}{p} - \frac{1}{r+1}\right)\int_\Omega |Du|^p + \left(\frac{1}{r+1} - \frac{1}{q}\right)\int_\Omega |x|^\sigma |u|^q, \\ &= \left(\frac{1}{p} - \frac{1}{q}\right)\int_\Omega |x|^\sigma |u|^q + \lambda\left(\frac{1}{p} - \frac{1}{r+1}\right)\int_\Omega u^{r+1}, \end{aligned} \tag{17}$$

which becomes coercive if $q \geq p$ and $p \leq r+1$.

Lemma 8. *Suppose $q \geq p$ and $p \leq r+1$, then Nehari manifold $\mathcal{N}$ is not empty.*

Proof. If $p = q$ or $p = r+1$ the result is obvious, suppose $q > p$ and $p < r+1$. Take any $u \in \mathbf{W}_0^{1,p}(\Omega) - \{0\}$ and consider the function given by

$$\begin{aligned} t \mapsto I'(tu)(tu) &= t^p \int_\Omega |Du|^p - t^q \int_\Omega |x|^\sigma |u|^q - t^{r+1}\lambda \int_\Omega u^{r+1} \\ &= t^{p-1}\left(t\int_\Omega |Du|^p - t^{q-p+1}\int_\Omega |x|^\sigma |u|^q - t^{r+2-p}\lambda \int_\Omega u^{r+1}\right) \\ &:= t^{p-1} z(t) \end{aligned}$$

The function $z(t)$ satisfies:

$$\begin{aligned} z'(t) &= \|Du\|_p^p - (q-p+1)t^{q-p}N(u) - (r+2-p)t^{r+1-p}\lambda \int_\Omega u^{r+1} \\ z''(t) &= -(q-p+1)(q-p)t^{q-p-1}N(u) - (r+2-p)(r+1-p)t^{r-p}\lambda \int_\Omega u^{r+1} \end{aligned}$$

Notice that at $t = 0$, $z'(t) > 0$. On the other hand, we have $\lim_{t\to\infty} z'(t) = -\infty$. It follows from the intermediate value theorem that we can find $\tilde{t} \neq 0$ such that $z'(\tilde{t}) = 0$. Since, $z''(t) < 0$ everywhere, it follows that $z(t)$ has a global maximum at $t = \tilde{t}$. Additionally, we can't have $z(\tilde{t}) < 0$ since $z(0) = 0$.

We conclude that there exists $\bar{t} \in (\tilde{t}, \infty)$ such that $z(\bar{t}) = 0$. Therefore,

$$I'(\bar{t}u)(\bar{t}u) = 0,$$

hence $\bar{t}u \in \mathcal{N}$. □

Lemma 9. *If $2 \leq q \leq \frac{p(N+\sigma)}{N-p}$ and $\max\{-N, -q\} < \sigma < 0$, the functional $I(u)$ achieves a positive minimum when restricted to $\mathcal{N}$, i.e. there exists a positive function $v \in \mathbf{W}_0^{1,p}(\Omega)$ such that $I(v) = \min_{u\in\mathcal{N}} I(u)$.*

Proof. Consider the value

$$m := \min_{u\in\mathcal{N}} I(u)$$

First notice, that $m > 0$. Indeed, for $u \in \mathcal{N}$, using CKN inequality (5) and Sobolev's inequality (which itself is included in CKN):

$$\|Du\|_p^p = \int_\Omega |u|^\sigma |u|^q + \lambda \int_\Omega u^{r+1} \leq C\|Du\|_p^q + C'\|Du\|_p^{r+1}$$

It follows that if $u \in \mathcal{N}$ then $\|Du\|_p \geq C$ for some $C > 0$, which implies that $m > 0$.

Let $u_n \in \mathcal{N}$ be a minimizing sequence which we may assume is nonnegative $u_n \geq 0$.

Over the Nehari manifold $\mathcal{N}$, $I(u)$ is coercive, hence u_n is bounded and up to a subsequence $u_n \rightharpoonup v$ for some $v \in \mathbf{W}_0^{1,p}(\Omega)$. Using the weak lower semi-continuity of the norm, we have:

$$I(v) \leq \liminf I(u_n) = m.$$

We are left to prove that $v \in \mathcal{N}$. By hypothesis, we have:

$$\|Du_n\|_p^p = \int_\Omega |x|^\sigma |u_n|^q + \lambda \|u_n\|_{r+1}^{r+1}$$

Taking the lim inf on both sides, we get

$$\|Du\|_p^p \leq \int_\Omega |x|^\sigma |u|^q + \lambda \|u\|_{r+1}^{r+1}$$

If we have equality the proof is complete. Suppose by contradiction that we have a strict inequality. Take $t > 0$ such that $tu \in \mathcal{N}$, then:

$$0 < m \leq I(tu) \leq t^{r+1} I(u) \leq t^{r+1} \liminf I(u_n) \leq t^{r+1} m$$

We claim that we may assume $t < 1$ and reach a contradiction. Indeed, recall the definition of $z(t)$ in the proof of lemma 8:

$$z(t) = t \int_\Omega |Du|^p - t^{q-p+1} \int_\Omega |x|^\sigma |u|^q - t^{r+2-p} \lambda \int_\Omega u^{r+1}$$

We have:

$$\begin{aligned} z(t) &< t \left(\int_\Omega |x|^\sigma |u|^q + \|u\|_{r+1}^{r+1} \right) - t^{q-p+1} \int_\Omega |x|^\sigma |u|^q - t^{r+2-p} \lambda \int_\Omega u^{r+1} \\ &= \left(t - t^{q-p+1} \right) \int_\Omega |x|^\sigma |u|^q + \left(t - t^{r+2-p} \right) \lambda \int_\Omega u^{r+1} \end{aligned}$$

If $t \geq 1$ we have $z(t) < 0$, thus not zero. We must have $0 < t < 1$, a contradiction. $\square$

Theorem 10. *Suppose* $1 < p < N$, $\max\{p, 2\} \leq q \leq \frac{p(N+\sigma)}{N-p}, p \leq r+1 \leq p^*, p < \max\{r+1, q\}$ *and* $\max\{-N, -q\} < \sigma < 0$*, then there is a positive solution to problem* (15)*.*

Proof. We claim that the function $v \in \mathbf{W}_0^{1,p}(\Omega)$ obtained in lemma 9 is the desired solution. Consider the function:

$$\phi(s,t) := \|Du\|_p^p - t^{q-p} \int_\Omega |x|^\sigma |u|^q - t^{r+1-p} \lambda \int_\Omega u^{r+1}$$

It's easy to see that $\phi \in C^1$ and $\phi(0,1) = 0$. Notice that

$$\phi_t(0,1) < 0$$

It follows from the implicit function theorem that there exists a C^1 curve $t(s)$ such that $\phi(s,t(s))=0$. Hence, the application $s\mapsto t(s)(v+su)$ defines a curve on $\mathcal{N}$ with the property that $0\mapsto v$. If we set $\gamma(s)=I(t(s)(v+su))$, it follows that $s=0$ is a local minimum for $\gamma(s)$, hence

$$0=\gamma'(0)=I'(v)[t'(0)v+u]=t'(0)I'(v)v+I'(v)u=I'(v)u$$

and we conclude that v is the desired solution. □

2.3 The case $\boldsymbol{\lambda=0}$

In this scenario, we have two cases two consider: $\sigma\geq 0$ and $\sigma<0$.

2.3.1 $\boldsymbol{\sigma\geq 0}$

Theorem 11. *Suppose $1<p<N, 2\leq q\leq p^*, \lambda=0$ and $\sigma\geq 0$, then (*) has a positive solution.*

Proof. The problem becomes equivalent to show that the infimum

$$m:=\inf_{\substack{u\in\mathbf{W}_0^{1,p}(\Omega)\\ u\neq 0}}\frac{\int_\Omega |Du|^p}{\left(\int_\Omega |x|^\sigma|u|^q\right)^{\frac{p}{q}}}$$

is achieved by some function $u\neq 0$.

Notice that, by Sobolev's inequality it follows that $m>0$. Let u_n be a minimizing sequence, which we may assume is nonnegative. The homogeneity of

$$Q(u):=\frac{\int_\Omega |Du|^p}{\left(\int_\Omega |x|^\sigma|u|^q\right)^{\frac{p}{q}}}$$

guarantees that we may assume $\int_\Omega |x|^\sigma|u_n|^q=1$, in particular, u_n is bounded in $\mathbf{W}_0^{1,p}(\Omega)$. Hence, $u_n\rightharpoonup u$ in $\mathbf{W}_0^{1,p}(\Omega)$ up to a subsequence.

Finally, the weak lower semi continuity of the norm implies that $Q(u)=m$. It follows that $u\in\mathbf{W}_0^{1,p}(\Omega)$ satisfies

$$\int_\Omega |Du|^{p-2}DuDv=m\int_\Omega |x|^\sigma|u|^{q-2}uv\ \text{ for every } v\in\mathbf{W}_0^{1,p}(\Omega),$$

this implies that we can find a nontrivial nonnegative (thus positive) solution to (*), namely $\tilde{u}:=\left(\frac{1}{m}\right)^{\frac{1}{q-p}}u$. □

Remark 1. *The case $p=2$ and $\sigma\geq 0$ is classical and we may consider, instead of $|u|^{q-2}u$, certain functions $f(u)$ with superlinear growth, see [6, Sec. 8.5.2].*

2.3.2 $\sigma < 0$

Let $\sigma = -\alpha$, with $\alpha > 0$. As before, the problem becomes equivalent to show that the minimum

$$m := \min_{\substack{u \in \mathbf{W}_0^{1,p}(\Omega) \\ u \neq 0}} \frac{\int_\Omega |Du|^p}{\left(\int_\Omega \frac{|u|^q}{|x|^\alpha}\right)^{\frac{p}{q}}}$$

is achieved by some function $u \neq 0$.

Recall, that by Caffarelli-Kohn-Nirenberg inequality (5):

$$\left(\int_\Omega \frac{|u|^q}{|x|^\alpha}\right)^{\frac{p}{q}} \leq C \int_\Omega |Du|^p$$

where we must have $q \leq \frac{p(N-\alpha)}{N-p}$.

Reasoning as in the proof of theorem 11, it follows that $m > 0$ and we conclude:

Theorem 12. *Suppose $1 < p < N, 2 \leq q \leq \frac{p(N+\sigma)}{N-p}, \lambda = 0$ and $\max\{-N, -q\} < \sigma < 0$, then (*) has a positive solution.*

Remark 2. *If we consider $p = 2$ in the theorem above, we recover the result from [15]. Moreover, the authors in [15] prove non existence results for q supercritical. For a summary of known results in the case of classical solutions and $p = 2$, see [9]. Notice that if $p \neq 2$, we can't use the bootstrap argument, typical of linear elliptic equations, even if the right side of (*) is zero, namely, not all p-harmonic functions are C^2 if $p \neq 2$, the optimal regularity is in fact $C^{1,\alpha}_{loc}$, whose proof can be find in [5].*

3 Concluding remarks and open questions

We have restricted our analysis to the case $N \geq 3$. However, some of the ideas presented here can be used to analyze problem (*) when $N = 1$ or 2, specially in the case $N = 2$.

If $N = 1$, the problem becomes finding a nontrivial solution to:

$$\begin{cases} -(|u'|^{p-2}u')' = |x|^\sigma |u|^{q-2}u + \lambda u^r & \text{in } (0,1), \\ u(0) = u(1) = 0 \end{cases} \tag{18}$$

The type and even definition of solution will depend whether or not $\sigma \geq 0$. Despite its simplicity, some questions remain open even when $\lambda = 0$, see [9] for a discussion on this topic.

Some of the results described here used the fact that $q \leq q(\sigma) := \frac{p(N+\sigma)}{N-p}$ due to the validity of the CKN inequality. It's then natural to ask: *Are there positive solutions in the cases considered here, but with q having supercritical growth?* By supercritical we mean $\frac{p(N+\sigma)}{N-p} < q \leq p^*$.

Another assumption used was that $1 < p < N$ in the p-laplacian. It's then natural to consider what happens when $p = 1$ or $p \geq N$. In the former case, the equation becomes singular when $Du = 0$, and we lose many properties when we consider the

Sobolev space $W_0^{1,1}(\Omega)$. Hence the methods developed here won't be useful to prove existence of nontrivial solutions when $p = 1$.

We haven't discussed when Ω is an unbounded domain. It would be interesting if the same arguments apply mutatis mutandis or if radical changes are necessary to prove equivalent results. Also interesting, would be the asymptotic analysis of solutions in punctured disks or annuli as treated in [4], since the presence of $|x|^\sigma$ would interact with both the source λu^r and the p-laplacian, the result of this interaction could be unpredictable. The case $p \geq N$ is also very interesting, we believe the ideas presented here may adapted to this case.

If we want to increase the sophistication a little, we could substitute the p-laplacian with the infinity laplacian, and analyze existence or nonexistence of viscosity solutions to the Dirichlet problem

$$\begin{cases} -\Delta_\infty u = |x|^\sigma |u|^{q-2}u + \lambda u^r & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{19}$$

Since this is a quasilinear equation of non-divergence type, a completely new approach would be necessary to prove existence of nontrivial solutions. Fundamental questions about the infinity laplace operator remain open, such as optimal regularity, which is known for the p-harmonic but not for infinity harmonic functions, see [7, 8] for known results on this.

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