An elliptic equation with power nonlinearity and degenerate coercivity

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Abstract

We discuss the existence and regularity of solutions to the following Dirichlet problem:

$$\begin{cases} -\mathrm{div}\left(\frac{Du}{(1+|u|)^\theta}\right) = -\mathrm{div}\left(|u|^\gamma E(x)\right) + f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\theta, \gamma > 0$. An interesting feature of this problem is the interplay between the two nonlinearities, the degeneracy and the power nonlinearity.

Keywords: quasilinear, elliptic equation, regularity, existence, degenerate

MSC Classification: 35J70 , 35J15 , 35B65 , 35A01

1 Introduction

In these notes we study existence and regularity of solutions to a class of elliptic problems whose basic model is

$$\begin{cases} -\mathrm{div}\left(\frac{Du}{(1+|u|)^\theta}\right) = -\mathrm{div}\left(|u|^\gamma E(x)\right) + f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\theta > 0$, $E(x)$ is a vector field and $f(x)$ a function in $L^m(\Omega)$ with $m \geq 1$.

More generally, we will focus on the following problem:

$$\begin{cases} -\mathrm{div}(a(x,u)Du) = -\mathrm{div}\left(|u|^\gamma E(x)\right) + f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{D}$$

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where γ is a fixed number, $a(x,s)$ is a Caratheodory function which satisfies, for a.e. $x \in \Omega$, any $s \in \mathbb{R}$:

$$\frac{\alpha}{(1+|s|)^{\theta}} \leq a(x,s) \leq \beta, \tag{C}$$

where α, β are positive constants.

The summability of $E(x)$ and $f(x)$ will vary and will be specified below.

Remark 1. *Condition (C) implies that the operator in problem (D) is not coercive, hence the usual $\mathbf{H}_0^1(\Omega)$-existence theory cannot be applied directly. Moreover, since $E(x)$ is not necessarily a potential, i.e. $E = Dv$, the equation is not always variational.*

Most of our results will assume $\theta + \gamma$ small, the case where γ is large and $\theta = 0$ has been recently described in [6]. Related results can also be found in [5, 13].

When $\theta > 1$, nonexistence results exist are described in [1]. The rationale behind all these results is that when the summability of $E(x)$ and $f(x)$ is high enough, bounded weak solutions tend to exist, whereas in low summability cases we can only get distributional solution lying in some Sobolev space $\mathbf{W}_0^{1,q}(\Omega)$, for some $1 < q < 2$, and in some cases, a smallness condition on the source $f(x)$ is also required.

We will look for two types of solutions:

A **weak solution**, sometimes also called a *finite energy* solution, is a function $u \in \mathbf{H}_0^1(\Omega)$ such that for $f \in L^{2_*}(\Omega)$, $|u|^{\gamma} E \in [L^2(\Omega)]^n$ and we have:

$$\int_{\Omega} a(x,u) Du D\varphi = \int_{\Omega} |u|^{\gamma} E(x) D\varphi + \int_{\Omega} f\varphi \quad \forall \varphi \in \mathbf{H}_0^1(\Omega). \tag{3}$$

A **distributional solution** is a function $u \in \mathbf{W}_0^{1,1}(\Omega)$ such that $f \in L^1(\Omega)$, $|u|^{\gamma} E \in [L^1_{loc}(\Omega)]^n$ and we have:

$$\int_{\Omega} a(x,u) Du D\varphi = \int_{\Omega} |u|^{\gamma} E(x) D\varphi + \int_{\Omega} f\varphi \quad \forall \varphi \in \mathcal{C}_0^{\infty}(\Omega). \tag{4}$$

Remark 2. *Notice that every weak solution $u \in \mathbf{H}_0^1(\Omega)$, if there is one, is a distributional solution by definition.*

As mentioned above, nonexistence can occur when $\theta > 1$. As a counterweight, in the last section of this paper we add a lower order term to problem (D) and are able to recover existence and regularity of solutions in this scenario as well.

Existence and regularity of solutions to quasilinear elliptic equations is an old and interesting problem. The literature is vast, specially for semilinear equations, see for example [9, 8, 10, 11, 3, 2, 4, 6], and for more comprehensive treatment see [12, 7].

Notation & Assumptions

- $\Omega \subset \mathbb{R}^N$ is a bounded domain and $N \geq 3$.
- The space $\mathbf{H}_0^1(\Omega)$ denotes the usual Sobolev space which is the closure of $\mathcal{C}_0^{\infty}(\Omega)$, smooth functions with compact support using the Sobolev norm.

- For $q > 1$, q' denotes the Holder conjugate, i.e. $\frac{1}{q} + \frac{1}{q'} = 1$, and q^* denotes the Sobolev conjugate, defined by $q^* = \frac{qN}{N-q} > q$.
- For $p > 1$, p_* denotes $(p^*)'$, in particular, $2_* = \frac{2N}{N+2}$.
- We will use the somewhat standard notation for Stampacchia's truncation functions (see [7]):

$$T_k(s) = \max\{-k, \min\{s, k\}\}, \quad G_k(s) = s - T_k(s), \quad \text{for } k > 0.$$

- The letter C will always denote a positive constant which may vary from place to place.
- The Lebesgue measure of a set $A \subseteq \mathbb{R}^N$ is denoted by $|A|$.
- The symbol $\rightharpoonup$ denotes weak convergence.

2 Proof of the results

Fix $n > 0$, let $f_n(x) = T_n(f(x))$ and $E_n(x) = T_n(E(x))$, the latter is the vector field obtained from $E(x)$ by truncating its components by n. Consider the truncated equation:

$$-\mathrm{div}(a(x, T_n(u_n))Du_n) = -\mathrm{div}\,(|T_n(u_n)|^\gamma E_n) + f_n(x)$$

A simple application of Schauder's fixed point theorem guarantee the existence of weak solution, i.e. a function $u_n \in \mathbf{H}_0^1(\Omega)$ satisfying

$$\int_\Omega a(x, T_n(u_n))Du_n D\varphi = \int_\Omega |T_n(u_n)|^\gamma E_n D\varphi + \int_\Omega f_n\varphi \quad \forall \varphi \in \mathbf{H}_0^1(\Omega). \tag{5}$$

Moreover, since the right hand side is bounded, classical regularity results imply that $u_n \in L^\infty(\Omega)$ as well.

The following lemma will be needed below.

Lemma A. *[7, Lem 6.2] Suppose $f \in L^1(\Omega)$, and set*

$$g(k) = \int_\Omega |G_k(f)|.$$

Suppose

$$g(k) \leq \beta |A_k|^\alpha$$

for some $\alpha > 1$ and $\beta > 0$, where $A_k = \{|f| > k\}$. Then $f \in L^\infty(\Omega)$ and

$$\|f\|_\infty \leq C\beta$$

for some $C = C(\alpha, \Omega)$.

2.1 When m, r are sufficiently large

In our first result below, we seek *finite energy solutions*, that is, bounded weak solutions $u \in \mathbf{H}_0^1(\Omega)$. As mentioned in the introduction, the majority of results of this type require high summability on the source. The theorem below confirms that claim and also requires an additional summability of the vector field $E(x)$ as well.

Theorem 1. *Suppose* (C) *holds with* $0 < \theta < 1$ *and* $0 < \gamma < 1$ *satisfying*

$$\theta + \gamma < 1 \text{ and } \gamma < \frac{2}{N-2}$$

and $E \in [L^{2r}(\Omega)]^N$, $f \in L^m(\Omega)$, *such that*

$$\begin{aligned} r &\geq \max\left\{\frac{N}{2-(N-2)\gamma}, \left(\frac{1}{\theta+\gamma}\right)'\right\}, \\ m &> \max\left\{\frac{N}{2}, \left(\frac{2}{\theta+1}\right)'\right\}, \end{aligned} \tag{6}$$

Then the Dirichlet problem (D) *has a bounded weak solution* $u \in \mathbf{H}_0^1(\Omega) \cap L^\infty(\Omega)$.

Proof. The proof is divided in two steps: We first prove that u_n is bounded in $\mathbf{H}_0^1(\Omega)$, and then we prove the boundedness in $L^\infty(\Omega)$.

Consider $\varphi = [(1+|u_n|)^{\theta+1} - 1]\mathrm{sgn}(u_n)$ as a test function in (5). We have

$$\alpha(\theta+1)\int_\Omega |Du_n|^2 \leq (\theta+1)\int_\Omega (1+|u_n|)^{\theta+\gamma}|E||Du_n| + \int_\Omega |f|(1+|u_n|)^{\theta+1}.$$

Applying Young's inequality to the above twice we obtain

$$\frac{\alpha(\theta+1)}{2}\int_\Omega |Du_n|^2 \leq \frac{1}{2\alpha}\int_\Omega (1+|u_n|)^{2(\theta+\gamma)}|E|^2 + \int_\Omega |f|(1+|u_n|)^{\theta+1}, \tag{7}$$

and

$$\int_\Omega |Du_n|^2 \leq \frac{1}{4}\left[\int_\Omega (1+|u_n|)^{2r'(\theta+\gamma)}\right] + C\left[\int_\Omega |E|^{2r} + \int_\Omega |f|^m\right] + \frac{1}{4}\left[\int_\Omega (1+|u_n|)^{m'(\theta+1)}\right], \tag{8}$$

Since $2r'(\theta+\gamma) \leq 2$ and $m'(\theta+1) \leq 2$, by Poincare inequality it follows that:

$$\int_\Omega |Du_n|^2 \leq C\left[1 + \int_\Omega |E|^{2r} + \int_\Omega |f|^m\right],$$

hence $\|u_n\|_{2^*} \leq C$ and up to a subsequence $u_n \rightharpoonup u$ in $\mathbf{H}_0^1(\Omega)$.

Now, we prove that u_n is bounded in $L^\infty(\Omega)$. Define

$$H(s) = \int_0^s \frac{1}{(1+|s|)^\theta}.$$

Set $A_k = \{x \in \Omega \,|\, H(u_n) > k\}$, taking $\varphi = G_k(H(u_n))$ as a test function in (5) we obtain

$$\alpha \int_{A_k} |DH(u_n)|^2 \leq \int_{A_k} |u_n|^\gamma |E||DH(u_n)| + \int_{A_k} |f| G_k(H(u_n))$$

Simplifying using Holder's inequality:

$$\alpha \int_{A_k} |DH(u_n)|^2 \leq \left(\int_{A_k} |u_n|^{2\gamma}|E|^2\right)^{\frac{1}{2}} \left(\int_{A_k} |DH(u_n)|^2\right)^{\frac{1}{2}} + \left(\int_{A_k} |f|^{2_*}\right)^{\frac{1}{2_*}} \left(\int_{A_k} |DH(u_n)|^2\right)^{\frac{1}{2}}$$

where we have used Poincare's inequality on the last term to the right. We conclude that

$$\alpha \left(\int_{A_k} |DH(u_n)|^2\right)^{\frac{1}{2}} \leq \left(\int_{A_k} |u_n|^{2\gamma}|E|^2\right)^{\frac{1}{2}} + \left(\int_{A_k} |f|^{2_*}\right)^{\frac{1}{2_*}}$$

Using Holder's inequality again:

$$\alpha \left(\int_{A_k} |DH(u_n)|^2\right)^{\frac{1}{2}} \leq \left(\int_{A_k} |u_n|^{2^*}\right)^{\frac{\gamma}{2^*}} \left(\int_{A_k} |E|^{2\left(\frac{2^*}{2\gamma}\right)'}\right)^{\frac{2^*-2\gamma}{22^*}} + \|f\|_m |A_k|^{\frac{m-2_*}{2_* m}},$$

Now, by Sobolev's inequality:

$$\left(\int_{A_k} |G_k(H(u_n))|^{2^*}\right)^{\frac{1}{2^*}} \leq C\left(\|E\|_{2r} |A_k|^{\frac{2r - \frac{22^*}{2^*-2\gamma}}{2r}\frac{2^*-2\gamma}{22^*}} + \|f\|_m |A_k|^{\frac{m-2_*}{2_* m}}\right),$$

Finally, recall that:

$$\int_{A_k} |G_k(H(u_n))| \leq \left(\int_{A_k} |G_k(H(u_n))|^{2^*}\right)^{\frac{1}{2^*}} |A_k|^{\frac{1}{2_*}}$$

Combing this with the condition (6), we conclude that by Lemma A, $\|H(u_n)\|_\infty \leq C$, but since $\lim_{s\to\pm\infty} H(s) = \pm\infty$, we deduce

$$\|u_n\|_\infty \leq C.$$

By the dominated convergence theorem, we can easily pass the limit in the first integral in (5). Similarly, we can pass the limit in the third integral, the only part not so obvious is the second integral.

Notice that given $M \subset \Omega$ measurable set:

$$\int_M (|T_n(u_n)|^\gamma E_n)\, D\varphi \leq C\|D\varphi\|_2 \|u_n\|_\infty^\gamma \left(\int_M |E|^2\right)^{\frac{1}{2}}$$

Therefore, the integral above is equi-integrable and the result follows from Vitali's convergence theorem. $\square$

2.2 Low summability of $E(x)$, i.e. $E \in [L^2(\Omega)]^N$

In our next result, we drop the summability assumption on $E(x)$ and as a result, finite energy solutions are not guaranteed to exist anymore and we can only hope for distributional solutions as the theorem below shows.

Theorem 2. *Suppose (C) holds with $0 < \theta < 1$ and $0 < \gamma < 1$ satisfying*

$$\theta + 2\gamma < 1,$$

$E \in [L^2(\Omega)]^N$, $f \in L^m(\Omega)$ with $m > \frac{q^*}{q^*-1+\theta+2\gamma}$, where $q = \frac{2N(1-\theta-\gamma)}{N-2(\theta+\gamma)}$. Then the Dirichlet problem (D) has a distributional solution $u \in \boldsymbol{W}_0^{1,q}$.

Proof. Set $\varphi = [(1+|u_n|)^\lambda - 1]\mathrm{sgn}(u_n)$ as a test function in (5), where $\lambda < 1$ will be specified later. We have

$$\alpha\lambda \int_\Omega (1+|u_n|)^{\lambda-1-\theta}|Du_n|^2 \leq \lambda \int_\Omega (1+|u_n|)^{\gamma+\lambda-1}|E||Du_n| + \int_\Omega |f||u_n|^\lambda$$

Using Young's inequality we obtain:

$$\alpha\lambda \int_\Omega (1+|u_n|)^{\lambda-1-\theta}|Du_n|^2 \leq \frac{1}{2\alpha}\int_\Omega |E|^2 + \frac{\alpha\lambda}{2}\int_\Omega (1+|u_n|)^{2(\gamma+\lambda-1)}|Du_n|^2 + \int_\Omega |f||u_n|^\lambda$$

Now choose $\lambda = 1-\theta-2\gamma$, simplifying we have:

$$\frac{\alpha(1-\theta-2\gamma)}{2}\int_\Omega \frac{|Du_n|^2}{(1+|u_n|)^{2(\theta+\gamma)}} \leq \frac{1}{2\alpha}\int_\Omega |E|^2 + \int_\Omega |f||u_n|^\lambda. \tag{9}$$

For any $q < 2$, by Holder's inequality using $\frac{2}{q}$ and $(\frac{2}{q})'$:

$$\begin{aligned} C\left(\int_\Omega |u_n|^{q^*}\right)^{\frac{q}{q^*}} &\leq \int_\Omega |Du_n|^q \leq \int_\Omega \frac{(1+|u_n|)^{q(\theta+\gamma)}|Du_n|^q}{(1+|u_n|)^{q(\theta+\gamma)}} \\ &\leq \left(\int_\Omega \frac{|Du_n|^2}{(1+|u_n|)^{2(\theta+\gamma)}}\right)^{\frac{q}{2}} \left(\int_\Omega (1+|u_n|)^{\frac{2q(\theta+\gamma)}{2-q}}\right)^{\frac{2-q}{2}} \end{aligned} \tag{10}$$

Combining this with (9):

$$\left(\int_\Omega |u_n|^{q^*}\right)^{\frac{2}{q^*}} \leq C\left(\int_\Omega (1+|u_n|)^{\frac{2q(\theta+\gamma)}{2-q}}\right)^{\frac{2-q}{q}} \left[\frac{1}{2\alpha\lambda}\int_\Omega |E|^2 + \int_\Omega |f||u_n|^\lambda\right]$$

Set $q = \frac{2N(1-\theta-\gamma)}{N-2(\theta+\gamma)}$, then $q^* = \frac{2q(\theta+\gamma)}{2-q}$ and $\frac{2}{q^*} > \frac{2-q}{q}$. We have:

$$\left(\int_\Omega |u_n|^{q^*}\right)^{\frac{2}{q^*}-\frac{2-q}{q}} \leq C\left[\int_\Omega |E|^2 + \int_\Omega |f|^m + \int_\Omega |u_n|^{\lambda m'}\right]$$

Since m satisfies $q^* = \lambda m'$ we obtain

$$\|u_n\|_{q^*} \leq C,$$

Notice that by (10) we also have:

$$\|Du_n\|_q \leq C.$$

It follows that u_n is bounded in $\mathbf{W}_0^{1,q}$ and up to subsequence $u_n \rightharpoonup u$.

By the dominated convergence theorem, we can easily pass the limit in the first integral in (5) if we assume $\varphi \in \mathcal{C}_0^\infty(\Omega)$. As before, the only nontrivial part is the second integral.

Notice that given $M \subset \Omega$ measurable set:

$$\int_M (T_n(u_n)^\gamma E_n)\, D\varphi \leq C\|D\varphi\|_\infty \|u_n\|_{q^*}^\gamma \left(\int_M |E|^{\frac{q^*}{q^*-\gamma}}\right)^{\frac{q^*-\gamma}{q^*}}$$

Therefore, the integral above is equi-integrable and the result follows from Vitali's convergence theorem. □

3 The presence of lower order term

In this last section we consider the effects on the existence and regularity of the presence of a lower order term in problem (D). More precisely, we consider

$$\begin{cases} -\operatorname{div}(a(x,u)Du) + u = -\operatorname{div}(|u|^\gamma E(x)) + f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{L}$$

where $\gamma > 0$ and $a(x,s)$ satisfies (C).

Similar to the previous case, for a fixed $n > 0$, Schauder's fixed point theorem can be used to guarantee the existence of weak solution $u_n \in \mathbf{H}_0^1(\Omega) \cap L^\infty(\Omega)$, satisfying

$$\int_\Omega a(x, T_n(u_n)) Du_n D\varphi + \int_\Omega u_n\varphi = \int_\Omega (|T_n(u_n)|^\gamma E_n)\, D\varphi + \int_\Omega f_n\varphi \quad \forall \varphi \in \mathbf{H}_0^1(\Omega). \tag{11}$$

We need the following lemma first:

Lemma B. *Suppose $\theta + 2\gamma < 2$, $E \in [L^{2r}(\Omega)]^N$ with $r \geq \frac{m}{2-2\gamma-\theta}$, $f \in L^m(\Omega)$, with $m \geq 2$. Then:*

$$\int_\Omega |u_n|^m \leq C \left(\int_\Omega |E|^{\frac{2m}{2-2\gamma-\theta}} + \int_\Omega |E|^2 + \int_\Omega |f|^m \right).$$

If $m = 1$, for any $\theta > 0, \gamma > 1$ and $r \geq 1$ we have:

$$\int_\Omega |u_n| \leq \int_\Omega |f|.$$

Proof. If $m = 1$, fix $k > 0$ and take $\varphi = \frac{T_k(u_n)}{k}$ as a test function. We have, ignoring the first positive term:

$$\int_\Omega u_n \frac{T_k(u_n)}{k} \leq k^{\gamma-1} \int_\Omega |E||Du_n| + \int_\Omega |f|$$

Taking the limit $k \to 0$ and using Fatou's lemma:

$$\int_\Omega |u_n| \leq \int_\Omega |f|$$

Fix $\lambda > 1$, take $\varphi = |1 + |u_n||^{\lambda-2}(1 + |u_n|)\mathrm{sgn}(u_n)$ as a test function to obtain:

$$\alpha(\lambda-1) \int_\Omega (1+|u_n|)^{\lambda-2-\theta}|Du_n|^2 + \int_\Omega |u_n|^\lambda \leq C \int_\Omega (1+|u_n|)^{\lambda-2+\gamma}|E||Du_n| + \int_\Omega |f||u_n|^{\lambda-1}$$

After using Young's inequality, that becomes:

$$\int_\Omega |u_n|^\lambda \leq C \int_\Omega (1 + |u_n|)^{2[(\lambda-2+\gamma)-\frac{\lambda-2-\theta}{2}]}|E|^2 + \int_\Omega |f||u_n|^{\lambda-1}$$

Simplifying:

$$\int_\Omega |u_n|^\lambda \leq \frac{1}{4} \int_\Omega |u_n|^{2r'[(\lambda-2+\gamma)-\frac{\lambda-2-\theta}{2}]} + C \left(\int_\Omega |E|^{2r} + \int_\Omega |E|^2 + \int_\Omega |f|^m \right) + \frac{1}{4} \int_\Omega |u_n|^{m'(\lambda-1)}$$

Choosing $\lambda = m$ and $r' = \frac{m}{m-2+2\gamma+\theta}$ we obtain:

$$\frac{1}{2} \int_\Omega |u_n|^m \leq C \left(\int_\Omega |E|^{\frac{2m}{2-2\gamma-\theta}} + \int_\Omega |E|^2 + \int_\Omega |f|^m \right) \tag{12}$$

□

3.1 m, r sufficiently large

As we shall see in the next theorem, the presence of a low order term increase the regularity of solutions. This fact was already noticed in some cases when $\theta = 0, \gamma > 1$, see [6]. Here we extend this analysis to the case $\theta > 1$.

Theorem 3. *Suppose $\theta > 1$ and $\theta + 2\gamma < 2$, $f \in L^m(\Omega)$ with $m \geq \max\{2, \frac{\theta N}{2}\}$, $E \in [L^p(\Omega)]^N$ such that $p > \frac{Nm}{m-2\gamma-\theta}$. Then the Dirichlet problem (L) has bounded weak solution $u \in \mathbf{H}_0^1(\Omega) \cap L^\infty(\Omega)$.*

Proof. Set $A_k = \{(1+|u_n|)^{\theta-1} > (1+k)^{\theta-1}\} = \{|u_n| > k\}$ and take

$$\varphi = \frac{1}{\theta-1} G_{(1+k)^{\theta-1}}((1+|u_n|)^{\theta-1})\mathrm{sgn}(u_n) =: G_{k,n}\mathrm{sgn}(u_n)$$

as a test function in (11). We have:

$$\alpha \int_{A_k} \frac{|Du_n|^2}{(1+|u_n|)^2} + \int_{A_k} |u_n| G_{k,n} \leq C \int_{A_k} (1+|u_n|)^{\gamma+\theta-2}|E||Du_n| + \int_{A_k} |f||G_{k,n}|$$

Notice that by Young's inequality:

$$\begin{aligned}\int_{A_k} |f|G_{k,n} &\leq C_\epsilon \int_{A_k} |f|^\theta + \frac{\epsilon}{\theta-1} \int_{A_k} [(1+|u_n|)^{\theta-1} - (1+k)^{\theta-1}][(1+|u_n|)^{\theta-1} - (1+k)^{\theta-1}]^{\frac{1}{\theta-1}} \\ &\leq C_\epsilon \int_{A_k} |f|^\theta + \epsilon C_\theta \int_{A_k} G_{k,n}|u_n|\end{aligned}$$

Taking $\epsilon = \frac{1}{C_\theta}$ and combining with the equations above we get:

$$\alpha \int_{A_k} \frac{|Du_n|^2}{(1+|u_n|)^2} \leq C \int_{A_k} (1+|u_n|)^{\gamma+\theta-2}|E||Du_n| + C_\epsilon \int_{A_k} |f|^\theta.$$

Using Young's inequality again:

$$\frac{\alpha}{2} \int_{A_k} \frac{|Du_n|^2}{(1+|u_n|)^2} \leq \frac{1}{2\alpha} \int_{A_k} (1+|u_n|)^{2(\gamma+\theta-1)}|E|^2 + C_\epsilon \int_{A_k} |f|^\theta,$$

Using lemma B with r such that $2r'(\gamma+\theta-1) = m$, i.e. $r = \frac{m}{m-2(\gamma+\theta-1)}$, we obtain:

$$\begin{aligned}\int_{A_k} \left| D\log\left(\frac{1+|u_n|}{1+k}\right)\right|^2 &\leq C\left(\int_{A_k} |E|^{2r}\right)^{\frac{1}{r}} + C\|f\|_m^\theta |A_k|^{\frac{m-\theta}{m}} \\ &\leq C\left(\|E\|_p^2 |A_k|^{\frac{p-2r}{p}} + \|f\|_m^\theta |A_k|^{\frac{m-\theta}{m}}\right)\end{aligned}$$

Finally, Sobolev's inequality give us:

$$\left(\int_{A_k} |\log(1+|u_n|)-\log(1+k)|^{2^*}\right)^{\frac{2}{2^*}} \leq C\left(\|E\|_p^2|A_k|^{\frac{p-2r}{p}}+\|f\|_m^{\theta}|A_k|^{\frac{m-\theta}{m}}\right)$$

We conclude that:

$$\int_{A_k} |\log(1+|u_n|)-\log(1+k)|^{2^*} \leq C\left(\|E\|_p^2|A_k|^{\frac{2^*(p-2r)}{2p}}+\|f\|_m^{\theta}|A_k|^{\frac{2^*(m-\theta)}{2m}}\right)$$

Since $m > \frac{\theta N}{2}$ implies $\frac{2^*(m-\theta)}{2m}, \frac{2^*(p-2r)}{2p} > 1$, lemma A gives $\|\log(1+|u_n|)\|_\infty \leq C$ and consequently:

$$\|u_n\|_\infty \leq C$$

It suffices now to choose any $n > C$ such that $T_n(u_n) = u_n$, for this particular n, u_n is a bounded weak solution of problem (L). $\square$

3.2 $2 \leq m < \theta + 2$

In our last result we slightly weaken the summability of the source $f(x)$, the cost of this is the existence of a distributional solution only, instead of a bounded weak solution.

Theorem 4. *Suppose $\theta+2\gamma < 2$, $f \in L^m(\Omega)$ with $2 \leq m < \theta+2$, and $E \in [L^{2r}(\Omega)]^N$ with $r \geq \max\left(\frac{m}{2-2\gamma-\theta}, \frac{m}{6+\theta-2m-2\gamma}\right)$. Then the Dirichlet problem (L) has distributional solution $u \in \boldsymbol{W}_0^{1,\frac{2m}{\theta+2}} \cap L^m(\Omega)$.*

Proof. Consider $\varphi = [(1+|u_n|)^{m-1}-1]\text{sgn}(u_n)$ as a test function. We have

$$\int_\Omega \frac{|Du_n|^2}{(1+|u_n|)^{\theta-m+2}} \leq C\left(\int_\Omega (1+|u_n|)^{\gamma+m-2}|E||Du_n| + \int_\Omega |f||u_n|^{m-1}\right)$$

Using Young's inequality, the fact that $m < \theta+2$, and lemma B again, we have:

$$\begin{aligned}\int_\Omega \frac{|Du_n|^2}{(1+|u_n|)^{\theta-m+2}} &\leq C\left(\int_\Omega (1+|u_n|)^{2[\gamma+m-2+\frac{\theta-m+2}{2}]}|E|^2 + \int_\Omega |f||u_n|^{m-1}\right)\\ &\leq C\left(\|E\|_{2r}^2 + \|f\|_m^m\right)\end{aligned}$$

We conclude that:

$$\int_\Omega \frac{|Du_n|^2}{(1+|u_n|)^{\theta-m+2}} \leq C$$

For any $q < 2$, by Holder's inequality using $\frac{2}{q}$ and $(\frac{2}{q})'$:

$$\int_\Omega |Du_n|^q = \int_\Omega \frac{(1+|u_n|)^{\frac{q(\theta-m+2)}{2}}}{(1+|u_n|)^{\frac{q(\theta-m+2)}{2}}}|Du_n|^q \leq C\left(\int_\Omega (1+|u_n|)^{\frac{q(\theta-m+2)}{2-q}}\right)^{\frac{2-q}{2}}$$

Set $q = \frac{2m}{\theta+2}$ then $\frac{q(\theta-m+2)}{2-q} = m$ and we conclude that

$$\int_\Omega |Du_n|^{\frac{2m}{\theta+2}} \leq C.$$

Therefore, $u_n \rightharpoonup u$ up to a subsequence and as before, u is a distributional solution. $\square$

4 Concluding remarks and open questions

Notice that we have assumed $N \geq 3$ in this manuscript due to some estimates failing when $N = 2$. It would be interesting to see if the arguments presented here can be adapted to include similar results in the plane as well. Hence it's reasonable to ask the following question:

What are the equivalent results of the ones presented here in 2 dimensions?

In this work the assumption $\gamma > 0$ was heavily used, so it would be interesting to see the equivalent results, if any, in the case $\gamma < 0$. Notice in this case the nonlinearity would compete with the degeneracy but this time also being a singularity so it's possible that no bounded solutions exists and if they do it's possible that some smallness condition will be required contrary to the case described here in theorem 1. We ask the following:

Is it still possible to obtain finite energy solutions if $\gamma < 0$ without smallness condition on the source or vector field E?

We can increase the level of difficulty of the Dirichlet problem (L) if instead of adding $u(x)$, we add $g(u)$ for some real valued function $g(s)$ with reasonable growth. It would be interesting to see if one can obtain Ambrosetti–Prodi type results in this case. More precisely, consider the problem:

$$\begin{cases} -\operatorname{div}(a(x,u)Du) + g(u) = -\operatorname{div}\left(|u|^\gamma E(x)\right) + f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{13}$$

Is it possible to find a function $g(s)$ Lipschitz with $g(0) = 0$ such that for any given source $f(x)$ only one of the following three options are possible: the above system has no solution, one solution, or two solutions.

References

[1] Angelo Alvino et al. “Existence results for nonlinear elliptic equations with degenerate coercivity”. In: *Annali di Matematica Pura ed Applicata* 182.1 (2003), pp. 53–79. DOI: https://doi.org/10.1007/s10231-002-0056-y.

[2] Antonio Ambrosetti and Paul H Rabinowitz. “Dual variational methods in critical point theory and applications”. In: *Journal of Functional Analysis* 14.4 (1973), pp. 349–381. ISSN: 0022-1236. DOI: https://doi.org/10.1016/0022-1236(73)90051-7. URL: https://www.sciencedirect.com/science/article/pii/0022123673900517.

[3] H. Berestycki and P. L. Lions. “Nonlinear scalar field equations, I existence of a ground state”. In: *Archive for Rational Mechanics and Analysis* 82.4 (1983), pp. 313–345. DOI: https://doi.org/10.1007/BF00250555. URL: https://doi.org/10.1007/BF00250555.

[4] Lucio Boccardo. “Some Developments on Dirichlet Problems with Discontinuous Coefficients”. eng. In: *Bollettino dell'Unione Matematica Italiana* 2.1 (Feb. 2009), pp. 285–297.

[5] Lucio Boccardo and Stefano Buccheri. “A nonlinear homotopy between two linear Dirichlet problems”. In: *Revista Matemática Complutense* 34.2 (2021), pp. 541–558. DOI: 10.1007/s13163-020-00363-x. URL: https://doi.org/10.1007/s13163-020-00363-x.

[6] Lucio Boccardo, Stefano Buccheri, and G. R. Cirmi. “Elliptic problems with superlinear convection terms”. In: *Journal of Differential Equations* 406 (2024), pp. 276–301. ISSN: 0022-0396. DOI: https://doi.org/10.1016/j.jde.2024.06.014. URL: https://www.sciencedirect.com/science/article/pii/S0022039624003863.

[7] Lucio Boccardo and Gisella Croce. *Elliptic Partial Differential Equations Existence and Regularity of Distributional Solutions*. Berlin, Boston: De Gruyter, 2013. ISBN: 9783110315424. DOI: https://doi.org/10.1515/9783110315424.

[8] Haïm Brezis and Shoshana Kamin. “Sublinear elliptic equations in $\mathbb{R}^n$”. In: *manuscripta mathematica* 74.1 (1992), pp. 87–106. DOI: 10.1007/BF02567660. URL: https://doi.org/10.1007/BF02567660.

[9] Wei-Yue Ding and Wei-Ming Ni. “On the existence of positive entire solutions of a semilinear elliptic equation”. In: *Archive for Rational Mechanics and Analysis* 91.4 (1986), pp. 283–308. DOI: 10.1007/BF00282336. URL: https://doi.org/10.1007/BF00282336.

[10] G. Díaz and R. Letelier. “Explosive solutions of quasilinear elliptic equations: existence and uniqueness”. In: *Nonlinear Analysis: Theory, Methods & Applications* 20.2 (1993), pp. 97–125. ISSN: 0362-546X. DOI: https://doi.org/10.1016/0362-546X(93)90012-H. URL: https://www.sciencedirect.com/science/article/pii/0362546X9390012H.

[11] D. G. Figueiredo, O. H. Miyagaki, and B. Ruf. “Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range”. In: *Calculus of Variations and Partial Differential Equations* 3.2 (1995), pp. 139–153. DOI: 10.1007/BF01205003. URL: https://doi.org/10.1007/BF01205003.

[12] David Gilbarg and Neil S Trudinger. *Elliptic partial differential equations of second order*. en. 3rd ed. Classics in mathematics. Berlin, Germany: Springer, 2001, p. 518. ISBN: 9783642617980. DOI: http://dx.doi.org/10.1007/978-3-642-61798-0.

[13] A. Marah and H. Redwane. “Existence Result for Solutions to Some Noncoercive Elliptic Equations”. In: *Acta Applicandae Mathematicae* 187.1 (2023), p. 18.

DOI: 10.1007/s10440-023-00609-y. URL: https://doi.org/10.1007/s10440-023-00609-y.