Exam I

Choose (only) 4 questions:

1. Give an example of three functions f, g, h such that $f \circ (g + h) \neq f \circ g + f \circ h$

   Proof. Take $f(x) = 1, g(x) = 1$ and $h(x) = 1$. $\square$

2. Find the largest natural number m such that $n^3 - n$ is divisible by m for all $n \in \mathbb{N}$. Prove your assertion.

   Proof. Notice that $n^3 - n = (n-1)n(n+1)$ is the product of three consecutive numbers, hence divisible by 6. We claim $m = 6$. Indeed, if $n = 1$ then $n^3 - n = 0$ which is divisible by 6. Suppose $n^3 - n$ is divisible by 6 then $(n+1)^3 - (n+1) = (n^3 - n) + 3n(n+1)$ is also divisible by 6, hence by induction 6 divides all the numbers of form $n^3 - n$, since 6 itself is one of those numbers, it is the maximum divisor. $\square$

3. Let $n \in \mathbb{N}$. Show that there is no $m \in \mathbb{N}$ such that $n < m < n + 1$.

   Proof. We use induction on $n \in \mathbb{N}$. It's true for $n = 1$, suppose valid for n. If there was a number m such that $n+1 < m < n+2$ then $n < m-1 < m+1$, a contradiction. $\square$

4. Prove the induction principle assuming the principle of well-ordering.

   Proof. Suppose the principle of well-ordering is true and $X \subseteq \mathbb{N}$ has the property that $1 \in X$ and $n \in X \Rightarrow n+1 \in X$. Suppose that $X \neq \mathbb{N}$, by the principle of well-ordering $\mathbb{N} - X$ has a minimum element, say m. Since $m \neq 1$, m is the successor of an element, say a, i.e. m=a+1, by minimality of m we must have $a \in X$, a contradiction since $a + 1 = m \notin X$. $\square$

5. Show that the set $P = \{\, n \in \mathbb{N}\,;\, n \text{ is prime} \,\}$ is infinite.

   Proof. Since $P \subseteq \mathbb{N}$, P is countable. Suppose $P = \{p_1, p_2, \ldots, p_m\}$ finite, then it's also bounded, so one p_i is the maximum, but then the number $p_1 \cdot p_2 \cdot \ldots \cdot p_m + 1$ would be greater than all p_i and not divisible by any of them, a contradiction. $\square$

6. Let Y be countable and $f : X \to Y$ such that $f^{-1}(y)$ is countable for each $y \in Y$. Show that X is countable.

   Proof. $X = \bigcup_{y \in Y} f^{-1}(y)$ is a countable union of countable sets, hence countable. $\square$

7. Given an example of $X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots$, an infinite sequence of nested **infinite** subsets such that

$$\bigcap_{i=1}^{\infty} X_i = \emptyset$$

Proof. Take $X_i = \{i, i+1, i+2, \ldots\}$. $\square$

8. Show that the set of all finite subsets of $\mathbb{N}$ is countable.

   Proof. Let $X = \{A \subset \mathbb{N}; A \text{ is finite}\}$ and $X_i = \{A \subset \mathbb{N}; |A| = i\}$. Then

   $$X = \bigcup_{i=1}^{\infty} X_i$$

   It's enough to show that X_i is countable for each $i \in \mathbb{N}$. Consider the injective function $f_i : X_i \to \mathbb{N}^i$, that associates to each subset A its elements in $\mathbb{N}^i$. This function is clearly injective, since $\mathbb{N}^i$ is countable, the result follows. $\square$