Real Analysis: Functions of a real variable

Genival da Silva[*]

October 11, 2024

Contents

[*]email: gdasilva@tamusa.edu, website: `www.gdasilvajr.com`

1 Sets

A **set** X is a collection of objects, also called the *elements* of the set. If 'a' is an element of X, we write $a \in X$. On the other hand, if 'a' isn't an element of X, we write $a \notin X$.

A set X is *well defined* when there is a rule that allows us to say if an arbitrary element 'a' is or isn't an element of X.

Example 1. *The set X of all right triangles is well-defined. Indeed, given any object 'a', if 'a' is not a triangle or doesn't have a right angle then $a \notin X$. If 'a' is a right triangle then $a \in X$.*

Example 2. *The set X of all tall people is not well-defined. The notion of 'tall' is not universally defined, hence given any element a we can't say if $a \in X$ or $a \notin X$.*

Usually one uses the notation

$$X = \{a,\ b,\ c,\ \ldots\}$$

to represent the set X whose elements are $a, b, c, \ldots$, and if a set has no elements we denote it by $\emptyset$ and call it the **empty set**.

The set of *natural numbers* $1, 2, 3, \ldots$ will be represented by

$$\mathbb{N} = \{1, 2, 3,\ \ldots\}$$

The set of *integers* will be represented by

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

The set of *rational numbers*, that is, fractions $\frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$, will be denoted by

$$\mathbb{Q} = \{\, \frac{a}{b} \,|\, a, b \in \mathbb{Z},\, b \neq 0 \,\}$$

The vast majority of sets in mathematics are not defined by specifying its elements one by one. What usually happens is a set being defined by some property its elements satisfy, i.e. if a has property P than $a \in X$, whereas if a doesn't have property P then $a \notin X$. One writes

$$X = \{a \,|\, a \text{ has property } P\}$$

For example, the set

$$X = \{a \in \mathbb{N} \,|\, a > 10\},$$

consists of all natural numbers bigger than 10.

Given two sets A, B, one says that A is a **subset** of B or that A is *included* in B (B *contains* A), represented by $A \subseteq B$, if every element of A is an element of B.

Example 3. *We have the obvious inclusion of sets:*

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}.$$

Example 4. *Let X be the set of all squares and Y be the set of all rectangles. Then $X \subseteq Y$, since every square is a rectangle.*

When one writes $X \subseteq Y$, it's possible that $X = Y$. In case $X \neq Y$, we say X is a *proper subset*, the notation $X \subsetneq Y$ is sometimes used to indicate that X is a proper subset of Y.

Notice that to write $a \in X$ is equivalent to say $\{a\} \subseteq X$. Also, by definition, it's always true that $\emptyset \subseteq X$ for every set X.

It's easy to see that the inclusion of sets has the following properties:

1. *Reflexive*, $X \subseteq X$ for every set X;
2. *Anti-symmetric*, if $X \subseteq Y$ and $Y \subseteq X$ then $X = Y$;
3. *Transitive*, if $X \subseteq Y$ and $Y \subseteq Z$ then $X \subseteq Z$.

It follows that two sets X and Y are the same if and only if $X \subseteq Y$ and $Y \subseteq X$, that is to say, they have the same elements.

Given a set X, we define the *power set* of X, $\mathcal{P}(X)$ as

$$\mathcal{P}(X) = \{A \,|\, A \subseteq X\}.$$

The set $\mathcal{P}(X)$ is the set of all subsets of X, in particular it's never empty, it has at least $\emptyset$ and X itself as elements.

Example 5. *Let* $X = \{1, 2, 3\}$ *then*

$$\mathcal{P}(X) = \{\,\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\,\}.$$

Notice that by using the Fundamental Counting Principle, any set with n elements has 2^n subsets. Therefore, the number of elements of $\mathcal{P}(X)$ is 2^n.

2 Operation with sets

We given two sets X and Y, one can build many other sets. For example, the **union** of X and Y, denoted by $X \cup Y$ is the of elements that are in X or Y, more precisely:

$$X \cup Y = \{\, a \,|\, a \in X \text{ or } a \in Y\,\}.$$

Similarly, the **intersection** of X and Y, denoted by $X \cap Y$ is the of elements that are common to both X and Y:

$$X \cap Y = \{\, a \,|\, a \in X \text{ and } a \in Y\,\}.$$

If $X \cap Y = \emptyset$, then X and Y are said to be *disjoint*.

Example 6. *Let* $X = \{a \in \mathbb{N} \,|\, a \leq 100\}$ *and* $Y = \{a \in \mathbb{N} \,|\, a > 50\}$ *then*

$$X \cup Y = \mathbb{N} \textit{ and } X \cap Y = \{a \in \mathbb{N} \,|\, 50 < a \leq 100\}$$

Example 7. *The sets* $X = \{a \in \mathbb{N} \,|\, a > 1\}$ *and* $Y = \{a \in \mathbb{N} \,|\, a < 2\}$ *are disjoint, i.e.* $X \cap Y = \emptyset$ *since there is no natural number between* 1 *and* 2.

The **difference** between X and Y, denoted by $X - Y$ is the set of elements that are in X but not in Y, more precisely:

$$X - Y = \{\, a \,|\, a \in X \text{ and } a \notin Y\,\}.$$

Given an inclusion of sets $X \subseteq Y$, the **complement** of X in Y is the set $Y - X$, the notation X^c sometimes is used if there is no confusion about who the set Y is.

Example 8. *Consider the sets* $X = \{a \in \mathbb{N} \,|\, a \text{ is even}\}$ *and* $Y = \mathbb{N}$*. Then* $X \subseteq Y$ *and* $X^c = \{a \in \mathbb{N} \,|\, a \text{ is odd}\}$*.*

Proposition 9. *Given sets* A, B, C, D *the following properties are true:*

1. $A \cup \emptyset = A$; $A \cap \emptyset = \emptyset$
2. $A \cup A = A$; $A \cap A = A$
3. $A \cup B = B \cup A$; $A \cap B = B \cap A$
4. $A \cup (B \cup C) = (A \cup B) \cup C$; $A \cap (B \cap C) = (A \cap B) \cap C$
5. $A \cup B = A \Leftrightarrow B \subseteq A$; $A \cap B = A \Leftrightarrow A \subseteq B$
6. *if* $A \subseteq B$ *and* $C \subseteq D$ *then* $A \cup C \subseteq B \cup D$ *and* $A \cap C \subseteq B \cap D$
7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
8. $(A^c)^c = A$
9. $(A \cup B)^c = A^c \cap B^c$; $(A \cap B)^c = A^c \cup B^c$

Proof. The last property, $(A \cup B)^c = A^c \cap B^c$, will be demonstrated below, the others are trivial or can be proved in a similar way.

We prove that $(A \cup B)^c \subseteq A^c \cap B^c$. Let $a \in (A \cup B)^c$, then $a \notin A \cup B$, in particular, $a \notin A$ and $a \notin B$, hence $a \in A^c \cap B^c$.

Conversely, take $a \in A^c \cap B^c$. Then $a \notin A$ and $a \notin B$, so $a \notin A \cup B$ and it follows that $a \in (A \cup B)^c$. □

An *ordered pair* (a, b) is formed by two objects a and b, such that for any other such pair (c, d):

$$(a, b) = (c, d) \Leftrightarrow a = c \text{ and } b = d.$$

The elements a and b are called *coordinates* of (a, b), a is the first coordinate and b the second one.

The **cartesian product** $X \times Y$ of two sets X and Y is the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$:

$$X \times Y = \{\, (x, y) \,|\, x \in X \text{ and } y \in Y \,\}.$$

Remark 1. *An ordered pair is not the same as a set, i.e.* $(a, b) \neq \{a, b\}$*. Notice that* $\{a, b\} = \{b, a\}$ *but* $(a, b) \neq (b, a)$ *in general.*

Example 10. *Consider the sets* $X = \{1, 2, 3\}$ *and* $Y = \{a, b\}$*, then*

$$X \times Y = \{\, (1, a), (1, b), (2, a), (2, b), (3, a), (3, b) \,\}.$$

3 Functions

A **function** $f : X \to Y$ consists of three components: a set X, *the domain*, a set Y, the *co-domain*, and a rule that associates each element $a \in X$ an unique element in $f(a) \in Y$, $f(a)$ is called the *value* of $f(x)$ at a, or the image of a under $f(x)$.

Another common notation to denote a function is $x \mapsto f(x)$. In this case the domain and codomain can be identified by the context.

Example 11. *The function $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = n + 1$ is called the successor function.*

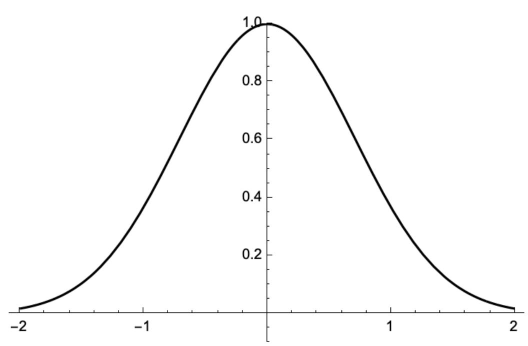
Example 12. *Let X be the set of all triangles. One can define a function $f : X \to \mathbb{R}$ by $f(x) =$ area of x.*

Example 13. *(Relation that is not a function) The correspondence that associates to each real number x, all y satisfying $y^2 = x$ is not a function because any $x \neq 0$ will be associated to two values, namely $\pm\sqrt{x}$, and in order to be a function every x has to have exactly one image $y = f(x)$.*

The graph of a function $f : X \to Y$ is a subset of $X \times Y$ defined by

$$\Gamma(f) = \{ (x, f(x)) \,|\, x \in X \}.$$

Example 14. *Consider the function $f(x) = e^{-x^2}$, its graph is given below:*

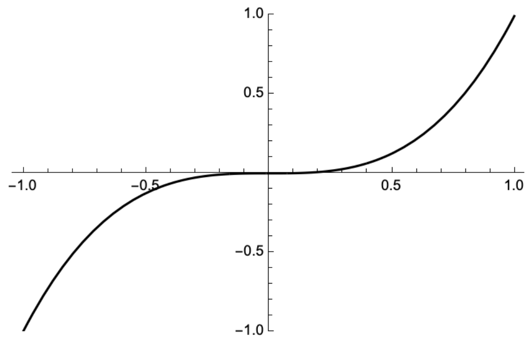


A function $f : X \to Y$ is said to be *injective or one-to-one* if for every x, y such that $f(x) = f(y)$ then $x = y$. Suppose $X \subseteq Y$, then inclusion $i : X \to Y$ given by $i(x) = x$ is a typical example of injective function.

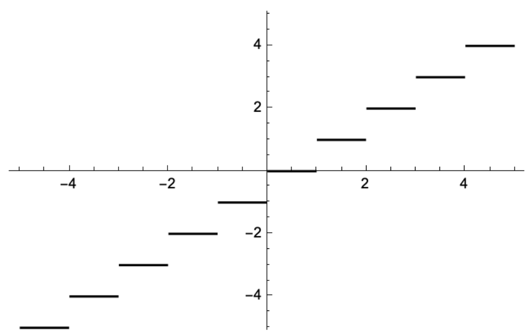
A function $f : X \to Y$ is said to be *surjective or onto* if for every $y \in Y$ there is $x \in X$ such that $y = f(x)$. The projection $p : X \times Y \to X$ in the first coordinate, given by $p(x, y) = x$ is a typical example of surjection.

Finally, a function $f : X \to Y$ is *bijective or a bijection* if it is both surjective and injective.

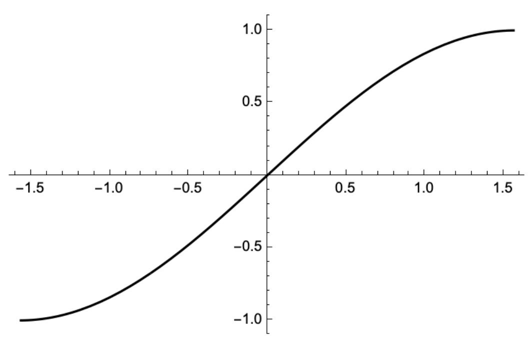
Example 15. *The function given by* $f(x) = x^3$ *is injective.*



Example 16. *The step function* $f(x) = \max\{\, n \in \mathbb{Z} \mid n \leq x \,\}$ *is not injective.*



Example 17. *The function* $f(x) = \sin x$ *is a bijection if we consider* $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$.



Given a function $f : X \to Y$, the *image of a set* $A \subseteq X$ is defined by

$$f(A) = \{\, y \in Y \mid y = f(a),\, a \in A \,\}.$$

Conversely, the *inverse image of a set* (sometimes called *pre-image*) $B \subseteq Y$ is given by

$$f^{-1}(B) = \{\, x \in X \mid f(x) \in B \,\}.$$

Proposition 18. *Given $f : X \to Y$ and subsets $A, B \subseteq X$, we have:*

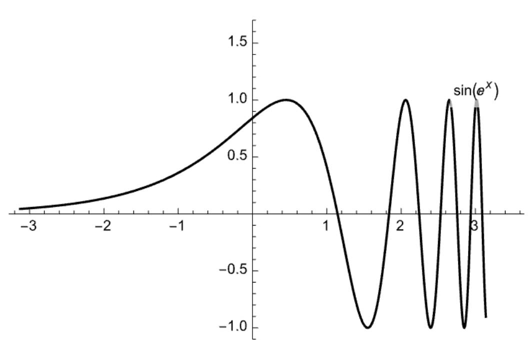
1. $f(A \cup B) = f(A) \cup f(B)$; $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
2. $f(A \cap B) \subseteq f(A) \cap f(B)$; $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$
3. *if* $A \subseteq B$ *then* $f(A) \subseteq f(B)$ *and* $f^{-1}(A) \subseteq f^{-1}(B)$
4. $f(\emptyset) = \emptyset$; $f^{-1}(\emptyset) = \emptyset$
5. $f^{-1}(Y) = X$
6. $f^{-1}(A^c) = (f^{-1}(A))^c$

Example 19. *Consider the function $f(x, y) = x^2 + y^2$, the inverse image $f^{-1}(\{1\})$ is a circle of radius 1. Similarly, any line $ax + by = c$ can be seen as $g^{-1}(\{c\})$, where $g(x, y) = ax + by$.*

Given two functions $f : X \to Y$ and $g : Y \to Z$, *the composition* $g \circ f$ of g and f is defined as the function:

$$(g \circ f)(x) = g(f(x))$$

Example 20. *The composition of the functions $g(x) = \sin x$ and $f(x) = e^x$ is the function $(g \circ f)(x) = \sin e^x$ depicted below.*



Given a function $f : X \to Y$ and a subset $A \subseteq X$, *the restriction* of $f(x)$ to A, denoted by $f|_A : A \to Y$, is defined by $f|_A(x) = f(x)$. Similarly, if $X \subseteq Z$, *a extension* of $f(x)$ to Z is any function $g : Z \to Y$ such that $g|_X(x) = f(x)$.

Example 21. *Consider again the function $f(x, y) = x^2 + y^2$, and the unit circle $\mathbb{S}^1 = \{ (x, y) \mid x^2 + y^2 = 1 \}$. Then the restriction $f|_{\mathbb{S}^1}$ is the constant function $g(x) = 1$.*

Given functions $f : X \to Y$, and $g : Y \to X$, the function $g(x)$ is called *left-inverse* of $f(x)$ if

$$(g \circ f)(x) = x.$$

Similarly, the function $g(x)$ is called *right-inverse* of $f(x)$ if

$$(f \circ g)(x) = x.$$

Finally, if there is a function $f^{-1}(x)$ such that

$$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x,$$

$f^{-1}(x)$ is called the *inverse* of $f(x)$. Notice that any inverse, if exists, is unique. If $g(x)$ and $h(x)$ are both inverses of $f(x)$ then

$$g(x) = g(f(h(x))) = (g \circ f)(h(x)) = h(x).$$

Proposition 22. *A function $f : X \to Y$ has an inverse $f^{-1} : Y \to X \Leftrightarrow f$ is bijective.*

Proof. Suppose f has an inverse f^{-1} and $f(x) = f(y)$ for some x, y. Taking the inverse on both sides, we conclude that $x = y$ and f is injective. Similarly, take $y \in Y$ and set $x = f^{-1}(y)$, then $f(x) = y$ and it follows that f is surjective.

Conversely, suppose f bijective. If $f(x) = y$, set $f^{-1}(y) = x$. One can easily check that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$. □

Example 23. *Consider the function $f : (0, +\infty) \to (0, +\infty)$ given by $f(x) = \frac{1}{x}$, then the f is its own inverse, i.e. $(f \circ f)(x) = x$.*

4 The natural numbers $\mathbb{N}$

The natural numbers are built axiomatically. Start with a set $\mathbb{N}$, whose elements are called *natural numbers*, and a function $s : \mathbb{N} \to \mathbb{N}$, called the *successor function*. For any $n \in \mathbb{N}$, $s(n)$ is called the successor of n.

The function $s(n)$ satisfies the following axioms:

Axiom 1. $s(n)$ is injective, i.e. every number has a unique successor.

Axiom 2. The set $\mathbb{N} - s(\mathbb{N})$ has only one element, which will be denoted by 1, i.e. every number has a successor and 1 is not a successor of any number.

Axiom 3. (Principle of induction) Let $X \subseteq \mathbb{N}$ be a subset with the following property: $1 \in X$ and given $n \in X$, $s(n) \in X$ as well. Then $X = \mathbb{N}$.

Whenever axiom 3 is used to prove a result, the result is said to be proved by induction.

Proposition 24. *For any $n \in \mathbb{N}$, $s(n) \neq n$.*

Proof. The proof is by induction. Let $X \in \mathbb{N}$ be a subset defined by:

$$X = \{\, n \in \mathbb{N} \,|\, s(n) \neq n \,\}.$$

By Axiom 2, $1 \in X$. Let $n \in X$, then $s(n) \neq n$. By Axiom 1, $s(s(n)) \neq s(n)$, hence $s(n) \in X$. The proof follows by Axiom 3.

□

Given a function $f : X \to X$, its power f^n is defined inductively. More precisely, if one sets $f^1 = f$ then f^n is defined by:

$$f^{s(n)} = f \circ f^n.$$

In particular, if one sets $2 = s(1), 3 = s(2), \ldots,$ then $f^2 = f \circ f, f^3 = f \circ f \circ f, \ldots$.

Now, given two natural numbers $m, n \in \mathbb{N}$, their sum $m+n \in \mathbb{N}$ is defined by:

$$m + n = s^n(m).$$

It follows that $m + 1 = s(m)$ and $m + s(n) = s(m + n)$, in particular:

$$m + (n + 1) = (m + n) + 1$$

More generally, the following can be proved using induction:

Proposition 25. *For any* $m, n, p \in \mathbb{N}$*:*

1. *(Associativity)* $m + (n + p) = (m + n) + p$*;*
2. *(Commutativity)* $m + n = n + m$*;*
3. *(Cancellation Law)* $m + n = m + p \Rightarrow n = p$*;*
4. *(Trichotomy) Only one of the following can occur:* $m = n$*, or* $\exists q \in \mathbb{N}$ *such that* $m = n + q$*, or* $\exists r \in \mathbb{N}$ *such that* $n = m + r$*.*

The notion of order among natural numbers can be defined in terms of addition. Namely, one writes

$$m < n,$$

if $\exists q \in \mathbb{N}$ such that $n = m + q$; in the same situation, one also writes $n > m$. Notice in particular that for every $m \in \mathbb{N}$:

$$m < s(m).$$

Finally, one writes $m \geq n$ if $m > n$ or $m = n$ and a similar definition applies to $\leq$.

Proposition 26. *For any* $m, n, p \in \mathbb{N}$*:*

(I) *(Transitivity)* $m < n, n < p \Rightarrow m < p$*;*

(II) *(Trichotomy) Only one of the following can occur:* $m = n$*,* $m < n$ *or* $m > n$*.*

(III) $m < n \Rightarrow m + p < n + p$*.*

The multiplication operation $m \cdot n$ will be defined in a similar way as $m+n$ was defined. Let $a_m : \mathbb{N} \to \mathbb{N}$ be the 'add m' function, $a_m(n) = n+m$. Then multiplication of two natural numbers $m \cdot n$ is defined as:

$$m \cdot 1 := m,$$
$$m \cdot (n+1) := (a_m)^n(m).$$

So $m \cdot 2 = a_m(m) = m+m, m \cdot 3 = (a_m)^2(m) = m+m+m, \ldots$, and it follows that:

$$m \cdot (n+1) := m \cdot n + m.$$

More generally, the following is true:

Proposition 27. *For any $m, n, p \in \mathbb{N}$:*

a. $m \cdot (n \cdot p) = (m \cdot n) \cdot p$*;*

b. $m \cdot n = n \cdot m$*;*

c. $m \cdot n = p \cdot n \Rightarrow m = p$*;*

d. $m \cdot (n + p) := m \cdot n + m \cdot p$*;*

e. $m < n \Rightarrow m \cdot p < n \cdot p$*.*

5 Well-ordering principle

Let $X \subseteq \mathbb{N}$. A number $m \in X$ is called **the minimum element** of X, denoted $m = \min X$, if $m \leq n$ for every $n \in X$. For example, 1 is the minimum of $\mathbb{N}$; 100 is the minimum of $\{100, 1000, 10000\}$.

Lemma 28. *If $m = \min X$ and $n = \min X$ then $m = n$.*

Proof. Since $m \leq p$ for every $p \in X$, $m \leq n$ in particular. Similarly, $n \leq m$ and hence $m = n$. □

The maximum element is defined similarly: $m = \max X$ if $m \geq n$, $\forall n \in X$. Notice that not all subsets $X \subseteq \mathbb{N}$ have a maximum. In fact, $\mathbb{N}$ itself doesn't have a maximum, since $m < m + 1$ for every $m \in \mathbb{N}$. The lemma above remains valid if we exchange 'minimum' by 'maximum'.

Despite not all subsets of $\mathbb{N}$ having a maximum, they do have a minimum if they are non-empty.

Theorem 29. *(Well-ordering principle) Let $X \subseteq \mathbb{N}$ be non-empty. Then X has a minimum.*

Proof. If $1 \in X$ then 1 is the minimum, so suppose $1 \notin X$. Let

$$I_n = \{\, m \in \mathbb{N} \,|\, 1 \leq m \leq n \,\},$$

and consider the set

$$L = \{\, n \in \mathbb{N} \,|\, I_n \subseteq X^c \,\}.$$

Since $1 \notin X \Rightarrow 1 \in L$. If $n \in L \Rightarrow n + 1 \in L$ then induction would imply $L = \mathbb{N}$, but $L \neq \mathbb{N}$ since $L \subseteq X^c = \mathbb{N} - X$, and $X \neq \emptyset$. We conclude that there is a m_0 such that $m_0 \in L$ but $m_0 + 1 \notin L$. It follows than $m_0 + 1$ is the minimum element of X. □

Corollary 30. *(Strong induction) Let $X \subseteq \mathbb{N}$ be a set with the following property:*

$$\forall n \in \mathbb{N}, \text{ if } X \text{ contains all } m < n \Rightarrow n \in X.$$

Then $X = \mathbb{N}$.

Proof. Set $Y = X^c$, the claim is that $Y = \emptyset$. Suppose not, that is, $Y \neq \emptyset$. By the theorem above, Y has a minimum element, say $p \in Y$. But then by hypothesis $p \in X$, a contradiction. □

Example 31. *Strong induction can be used to prove the* ***Fundamental theorem of Arithmetic****, which says that every number greater than 1 can written as a product of primes (a number p is* ***prime*** *if $p \neq m \cdot n$, with $m < p$ and $n < p$). Indeed, Let $X = \{ m \in \mathbb{N} \mid m \text{ is a product of primes} \}$ and $n \in \mathbb{N}$ a given number. If X contains all numbers m such that $m < n$, then if n is prime, $n \in X$; if n is not a prime then $n = p \cdot q$ with $p < n, q < n$, again it follows that $n \in X$. Therefore, strong induction implies $X = \mathbb{N}$.*

Let X be any set. A common way of defining a function $f : \mathbb{N} \to X$ is **by recurrence** (sometimes 'by induction' is used), namely, $f(1)$ is given and also a rule that allows one to obtain $f(m)$ knowing $f(n)$ for all $n < m$. Technically, more than one function f could exist satisfying these conditions, however it is know that such a function is unique, the proof of this fact is left as an exercise.

Example 32. *(Factorial) The factorial function $f : n \mapsto n!$ can be defined using induction. Set $f(1) = 1$ and $f(n+1) = (n+1) \cdot f(n)$. Then $f(2) = 2 \cdot 1$, $f(3) = 3 \cdot 2 \cdot 1$, $\ldots$, $f(n) = n!$.*

Example 33. *(Arbitrary sums/products) So far the definition of $m + n$ was used, what about $m + n + p$ or $m_1 + \ldots + m_n$? In order to define arbitrary sums (or products), one can use induction. Namely,*

$$m_1 + \ldots + m_n = (m_1 + \ldots + m_{n-1}) + m_n;$$

and similarly, for products:

$$m_1 \cdot \ldots \cdot m_n = (m_1 \cdot \ldots \cdot m_{n-1}) \cdot m_n.$$

6 Finite and Infinite sets

Throughout this section, I_n stands for the set of numbers less than or equal to n:

$$I_n = \{\, m \in \mathbb{N} \,|\, 1 \leq m \leq n \,\}$$

A arbitrary set X is **finite** if $X = \emptyset$ or there is number $n \in \mathbb{N}$ and a bijection

$$f : I_n \to X.$$

In the latter case, one says that X has n elements and writes:

$$|X| = n,$$

f is said to be a counting function for X. By convention, if $X = \emptyset$ then one says X has zero elements, i.e. $|\emptyset| = 0$.

It remains to show that the number of elements is a well-defined notion, that is to say, if there are bijections $f : I_n \to X$ and $g : I_m \to X$ then $n = m$.

Theorem 34. *Let $X \subseteq I_n$. If there is a bijection $f : I_n \to X$, then $X = I_n$.*

Proof. The proof is by induction on n. The case $n = 1$ is obvious, suppose the result true for n, the proof follows if one can prove the result for $n + 1$.

Suppose $X \subseteq I_{n+1}$ and there is a bijection $f : I_{n+1} \to X$. Let $a = f(n+1)$ and consider the restriction $f : I_n \to X - \{a\}$.

If $X - \{a\} \subseteq I_n$ then $X - \{a\} = I_n$, $a = n + 1$ and $X = I_{n+1}$.

Suppose $X - \{a\} \nsubseteq I_n$, then $n + 1 \in X - \{a\}$ and one can find b such that $f(b) = n + 1$. Let $g : I_{n+1} \to X$ be the defined by $g(m) = f(m)$ if $m \neq n + 1, a$; $g(n + 1) = n + 1$; $g(b) = a$. By construction, the restriction $g : I_n \to X - \{n + 1\}$ is a bijection and obviously $X - \{n + 1\} \subseteq I_n$, hence $X - \{n + 1\} = I_n$ and it follows that $X = I_{n+1}$. □

Corollary 35. *(Number of elements is well-defined) If there is a bijection $f : I_n \to I_m$ then $m = n$. Therefore, if $f : I_n \to X$ and $g : I_m \to X$ are bijections then $n = m$.*

Proof. The first part follows directly from the theorem. For the second part, consider the composition $(f^{-1} \circ g) : I_m \to I_n$. □

Corollary 36. *There is no bijection $f : X \to Y$ between a finite set X and a proper subset $Y \subseteq X$.*

Proof. By definition there is a bijection $\varphi : I_n \to X$ for some $n \in \mathbb{N}$. Since Y is proper, $A := \varphi^{-1}(Y)$ is also proper in I_n. Let $\varphi_A : A \to Y$ be the restriction of φ from I_n to A. Suppose there is a bijection $f : X \to Y$, then the composite function $\varphi_A^{-1} \circ f \circ \varphi : I_n \to A$ defines a bijection, a contradiction. □

Theorem 37. *Let X be a finite set and $Y \subseteq X$, then Y is finite and $|Y| \leq |X|$, the equality occurs only if $X = Y$.*

Proof. It's enough to prove the result for $X = I_n$. If $n = 1$ the result is obvious. Suppose the result is valid for I_n and consider $Y \subseteq I_{n+1}$. If $Y \subseteq I_n$, the induction hypothesis gives the result, so assume $n+1 \in Y$. Then $Y - \{n+1\} \subseteq I_n$ and by induction, there is a bijection $f : I_p \to Y - \{n+1\}$, where $p \leq n$. Let $g : I_{p+1} \to Y$ be a bijection defined by $g(n) = f(n)$ if $n \in I_n$, and $g(p+1) = n+1$. This proves that Y is finite, moreover since $p \leq n \Rightarrow p+1 \leq n+1$, $|Y| \leq n$. The last statement says that if $Y \subseteq I_n$ and $|Y| = n$ then $Y = I_n$, but this is a direct consequence of theorem 34. □

The following Corollary is immediate:

Corollary 38. *Let Y be finite and $f : X \to Y$ be an injective function. Then X is also finite and $|X| \leq |Y|$.*

Corollary 39. *Let X be finite and $f : X \to Y$ be an surjective function. Then Y is also finite and $|Y| \leq |X|$.*

Proof. Since f is surjective, by the proof of proposition 22, f has an injective right-inverse $g : Y \to X$. The result follows by the corollary above. □

A set X that is not finite is said to be **infinite**. More, precisely X is infinite when it's not empty and there is no bijection $f : I_n \to X$ for any $n \in \mathbb{N}$.

Example 40. *The natural numbers $\mathbb{N}$ is an infinite set since there is no surjection between I_n and $\mathbb{N}$, because given any function $f : I_n \to \mathbb{N}$, the number $f(1) + f(2) + \ldots + f(n)$ is not in the range.*

Example 41. *$\mathbb{Z}$ and $\mathbb{Q}$ are also infinite sets since they contain $\mathbb{N}$, which is infinite.*

A set $X \subseteq \mathbb{N}$ is **bounded**, if there is a number $M \in \mathbb{N}$ such that $n \leq M$ for all $n \in X$.

Theorem 42. *Let $X \subseteq \mathbb{N}$ be nonempty. The following are equivalent:*

a. *X is finite;*

b. *X is bounded;*

c. *X has a greatest element.*

Proof. The proof is based on the implications a $\Rightarrow$ b, b $\Rightarrow$ c, c $\Rightarrow$ a.

(a $\Rightarrow$ b) Let $X = \{x_1, x_2, \ldots, x_n\}$. Then $M = x_1 + \ldots + x_n$ satisfies $n \leq M$ for all $n \in X$.

(b $\Rightarrow$ c) Consider the set $A = \{\, n \in \mathbb{N} \,|\, n \geq x, \forall x \in X \,\}$. Since X is bounded, $A \neq \emptyset$. By the principle of well ordering, A has a minimum element, say $m \in A$. If $m \in X$ then m is the greatest element, so suppose $m \notin X$. By definition, $m > n$ for all $n \in X$, and since $X \neq \emptyset$, $m > 1$, that is $m = p + 1$, for some $p \in \mathbb{N}$. If $p \geq x$ for all $x \in X$ then $p \in A$, a contradiction since $p < m$ and m is minimal. If there is a $x \in X$ such that $x > p$, then $x \geq m$ a contradiction unless $x = m$, but $m \notin X$ by assumption. It follows that $m \in X$ and m is the greatest element.

(c $\Rightarrow$ a) If X has a greatest element, say M, then $X \subseteq I_M$ and it follows that X is finite.

□

The Theorem below follows directly from the definitions, the proof will be omitted.

Theorem 43. *Let X and Y be two sets such that $|X| = m, |Y| = n$ and $X \cap Y = \emptyset$. Then $X \cup Y$ is finite and $|X \cup Y|$=$m + n$.*

The following corollary is immediate:

Corollary 44. *Let $X_1, X_2, \ldots, X_n$, be a finite collection of sets such that each X_i is finite and $X_i \cap X_j = \emptyset$ if $i \neq j$. Then $\bigcup_{i=1}^{n} X_i$ is finite and*

$$|\bigcup_{i=1}^{n} X_i| = \sum_{i=1}^{n} |X_i|$$

.

Corollary 45. *Let $X_1, X_2, \ldots, X_n$, be a finite collection of sets such that each X_i is finite. Then $\bigcup\limits_{i=1}^{n} X_i$ is finite and*

$$|\bigcup_{i=1}^{n} X_i| \leq \sum_{i=1}^{n} |X_i|$$

.

Proof. For each $i = 1, \ldots, n$, set $Y_i = X_i \times \{i\}$. Then the projection

$$\pi_i : \bigcup_{i=1}^{n} Y_i \to \bigcup_{i=1}^{n} X_i$$

in the first coordinate is surjective, by Corollaries 39 and 44, the proof is complete. □

Corollary 46. *Let $X_1, X_2, \ldots, X_n$, be a finite collection of sets such that each X_i is finite. Then $X_1 \times \ldots \times X_n$ is finite and*

$$|X_1 \times \ldots \times X_n| = \prod_{i=1}^{n} |X_i|$$

.

Proof. It's enough to prove for $n = 2$, since the general case follows from this one. Let $X_2 = \{y_1, \ldots, y_m\}$, notice that $X_1 \times X_2 = X_1 \times \{y_1\} \cup \ldots \cup X_2 \times \{y_m\}$, the result follows by Corollary 44. □

7 Countable Sets

A set X is **countable** if it is finite or there is a bijection $f : \mathbb{N} \to X$. In the latter case, it is necessarily an infinite set, since as $\mathbb{N}$ is infinite, and we use the term **countably infinite**.

Example 47. *The set $X = \{\, 2n \in \mathbb{N} \,|\, n \in \mathbb{N} \,\}$ of all even numbers is countable. The function $f(x) = 2x$ defines a bijection between X and $\mathbb{N}$.*

Theorem 48. *Let X be an infinite set. Then X has a countably infinite subset.*

Proof. It's enough to find an injective function $f : \mathbb{N} \to X$, since every injective function is a bijection over its image. Choose an element $a_1 \in X$, set $X_1 = X - \{a_1\}$ and $f(1) = a_1$. Since X is infinite, X_1 is also infinite, choose an element a_2 in X_1, and set $f(2) = a_2$. Proceeding by induction, we have $f(n) = a_n$, $a_n \in X_{n-1}$, where $X_{n-1} = X - \{a_1, a_2, \ldots, a_{n-1}\}$.

Suppose $f(n) = f(m)$, with $n, m \in \mathbb{N}$, then $a_n = a_m$, which is possible only if $n = m$. Therefore, f is injective. □

Corollary 49. *A set X is infinite $\iff$ there is a bijection $f : X \to Y$, where $Y \subsetneq X$ is a proper subset.*

Proof. ($\Rightarrow$) Suppose X infinite, by theorem 48, X has a countably infinite subset, say $Z = \{a_1, a_2, a_3, \ldots\}$. Set $Y = (X - Z) \cup \{a_2, a_4, a_6, \ldots\}$ and define $f(x) = x$ if $x \in X - Z$, and $f(a_n) = a_{2n}$ otherwise. The function $f(x)$, defined this way, is clearly a bijection.

($\Leftarrow$) Follows from Corollary 36.

□

A function $f : X \to Y$ is called *increasing* if $x < y \Rightarrow f(x) < f(y)$.

Theorem 50. *Every subset X of $\mathbb{N}$ is countable.*

Proof. The proof is very similar to the one in theorem 48. If X is finite then is countable, so assume X infinite. We define an increasing bijection $f : \mathbb{N} \to X$ by induction. Let $X_1 = X$, $a_1 = \min X$ (which exists by Theorem 29), and set $f(1) = a_1$. Now, define $X_2 = X - \{a_1\}$ and $f(2) = a_2 = \min X_2$. By induction, we define $f(n) = a_n = \min X_n$, where $X_n = X - \{a_1, a_2, \ldots, a_{n-1}\}$. The function $f(n)$ is injective by construction, suppose $f(n)$ not surjective. There is $x \in X$ such that $x \notin f(\mathbb{N})$. So $x \in X_n$ for every n, which implies that $x > f(n)$ for every n, and x is a bound for the infinite set $f(\mathbb{N})$, a contradiction by Theorem 42. □

Corollary 51. *Let X be a countable set. Then for any $Y \subseteq X$, Y is countable.*

Corollary 52. *The set of all prime numbers is countable.*

Corollary 53. *Let Y be a countable set and $f : X \to Y$ an injective function. Then X is countable.*

Corollary 54. *The set $\mathbb{Z}$ of integers is countable.*

Proof. The function $f : \mathbb{Z} \to \mathbb{N}$ defined by $f(0) = 1, f(m) = 2m,$ if $m > 0$ and $f(m) = -2m + 1,$ if $m < 0$, is bijective. □

Corollary 55. *Let X be a countable set and $f : X \to Y$ a surjective function. Then Y is countable.*

Proposition 56. *The set $\mathbb{N} \times \mathbb{N}$ is countable.*

Proof. The function defined by $f(m, n) = 2^m 3^n$ is a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. □

Corollary 57. *Let $X_1, X_2, \ldots$ be a countable collection of countable sets. Set $X = \bigcup_{i=1}^{\infty} X_i$, then X is countable.*

Proof. Let $f_i : \mathbb{N} \to X_i$ be a counting function for each $i \in \mathbb{N}$. Then $f(i, m) := f_i(m)$ defines a surjection $f : \mathbb{N} \times \mathbb{N} \to X$. By Corollary 55, X is countable. □

Corollary 58. *If X, Y are countable sets then $X \times Y$ is countable.*

Proof. Let $f_1 : \mathbb{N} \to X, f_2 : \mathbb{N} \to Y$ be counting functions. Then $f(m, n) := (f_1(m), f_2(n))$ defines a bijection, Proposition 56 concludes the proof. □

Corollary 59. *The set $\mathbb{Q}$ of rational numbers is countable.*

Proof. Let $\mathbb{Z}^*$ denote the set of nonzero integers. Define the surjective function $f : \mathbb{Z} \times \mathbb{Z}^* \to \mathbb{Q}$ given by $f(m, n) = \frac{m}{n}$. By Corollary 55, $\mathbb{Q}$ is countable. □

8 Uncountable sets

A set X is **uncountable** if it's not countable. Given two sets X and Y, if there is a bijection $f : X \to Y$, we say X and Y have the same **cardinality**, in symbols:

$$\mathrm{card}(X) = \mathrm{card}(Y).$$

If we assume f injective only and there is no surjective function $g : X \to Y$, then we say

$$\mathrm{card}(X) < \mathrm{card}(Y).$$

The cardinality of the Natural numbers $\mathbb{N}$ is denoted by

$$\text{card}(\mathbb{N}) = \aleph_0.$$

If the set X is finite with n elements, we say $\text{card}(X) = n$. By definition, for any infinite set X:

$$\aleph_0 \leq \text{card}(X).$$

Recall that given two sets X and Y, the set $\mathcal{F}(X, Y)$ denotes the set of all functions betwenn X and Y.

Theorem 60. *(Cantor) Let X and Y be sets such that Y has at least two elements. There is no surjective function $\phi : X \to \mathcal{F}(X, Y)$.*

Proof. Suppose a function $\phi : X \to \mathcal{F}(X, Y)$ is given and let $\phi_x = \phi(x) : X \to Y$ be the image of $x \in X$, which itself is a function. We claim that there is a $f : X \to Y$ that is not ϕ_x for any X. Indeed, for each $x \in X$ let $f(x)$ be an element different than $\phi_x(x)$ (this is possible sice $|Y| \geq 2$), then $f \neq \phi_x$ for every $x \in X$ and hence, ϕ is not surjective. $\square$

Corollary 61. *Let $X_1, X_2, \ldots$ be a countable collection of countably infinite sets. Then the infinite cartesian product $X = \prod_{i=1}^{\infty} X_i$ is uncountable.*

Proof. It's enough to prove the result for $X_i = \mathbb{N}$. In this case, $X = \mathcal{F}(\mathbb{N}, \mathbb{N})$ and the result follows from Theorem 60. $\square$

Example 62. *The set $X = \{(a_1, a_2, a_3, a_4, \ldots\}$ of all sequence of natural numbers is uncountable.*

Example 63. *The set of all real numbers $\mathbb{R}$ is uncountable. This will be proved in the next sections.*

9 Fields

A **field** K is a set K together with two operations:

$$+ : K \times K \to K \text{ and } \cdot : K \times K \to K$$

satisfying the following properties (also called *field axioms*):

Given $x, y, z \in K$, we have:

1. $(x + y) + z = x + (y + z)$;

2. $x + y = y + x$;

3. There is an element $0 \in K$ such that $\forall x \in K$, $x + 0 = x$;

4. For any $x \in K$ there is an element $y \in K$ such that $x + y = 0$. We define $-x := y$, and write $z - x$ instead of $z + (-x)$;

5. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;

6. $x \cdot y = y \cdot x$;

7. There is an element $1 \in K$ such that $1 \neq 0$ and $\forall x \in K$, $x \cdot 1 = x$;

8. For any $x \neq 0$ there is an element $y \in K$ such that $x \cdot y = 1$. We define $x^{-1} := y$, and write $\frac{z}{x}$ instead of $z \cdot x^{-1}$;

9. $x \cdot (y + z) = x \cdot y + x \cdot z$.

Given two fields K and L, we say a function $f : K \to L$ is a *homomorphism*, if $f(x+y) = f(x)+f(y)$ and $f(c\cdot x) = c\cdot f(x)$. We say f is an *isomorphism* if, in addition, f is bijective and f^{-1} is also a homomorphism. An *automorphism* $f : K \to K$ is an isomorphism between K and itself.

Example 64. *The set rational numbers $\mathbb{Q}$ together with the operations*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{db} \text{ and } \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

is a field. In this case, $0 = \frac{0}{1}$, $1 = \frac{1}{1}$ and $(\frac{a}{b})^{-1} = \frac{b}{a}$.

Example 65. *If p is prime, the set of integers mod p, $\mathbb{Z}_p = \{\bar{0}, \ldots, \overline{p-1}\}$, with operations $\bar{a} + \bar{b} = \overline{a + b}$ and $\bar{a} \cdot \bar{b} = \overline{a \cdot b}$, is a field. It easy to see that $0 = \bar{0}, 1 = \bar{1}$ in this case. Moreover, by Fermat's little theorem $\bar{a} \cdot \bar{a}^{p-2} = \bar{1}$, hence $\bar{a}^{-1} = \bar{a}^{p-2}$.*

Example 66. *The set of rational functions, $\mathbb{Q}(t) = \{\, \frac{p(t)}{q(t)}\, ;\, p(t), q(t) \in \mathbb{Q}[t], q(t) \not\equiv 0\,\}$, where $\mathbb{Q}[t]$ is the set of polynomials with rational coefficients, with the usual operations of fractions is a field.*

Proposition 67. *Let K be a field and $x, y \in K$, then* *a.* $x \cdot 0 = 0$;

b. $x \cdot z = y \cdot z$ *and* $z \neq 0$ *then* $x = y$;

c. $x \cdot y = 0 \Rightarrow x = 0$ *or* $y = 0$;

d. $x^2 = y^2 \Rightarrow x = \pm y$.

Proof. a. Indeed, $x \cdot 0 + x = x \cdot (0 + 1) = x$, hence $x \cdot 0 = 0$.

b. We have $x = x \cdot z \cdot z^{-1} = y \cdot z \cdot z^{-1} = y$.

c. If $x \neq 0$ then $x \cdot y = 0 \cdot x \Rightarrow y = 0$.

d. Notice that $x^2 = y^2 \Rightarrow x^2 - y^2 = 0 \Rightarrow (x - y)(x + y) = 0$.

□

10 Ordered Fields

An ordered field is a field K together with a subset $P \subseteq K$, called the set of *positive elements*, such that for any $x, y \in P$ the following properties hold:

(I) *(Close under addition/multiplication)* $x + y \in P, x \cdot y \in P$;

(II) *(Trichotomy)* For any $x \in K$, only one of the following occurs: $x = 0$, $x \in P$,$-x \in P$.

If we denote $-P = \{\, -p\,;\, p \in P \,\}$, then K can be written as a disjoint union

$$K = P \cup -P \cup \{0\}$$

Notice that in an ordered field if $x \neq 0$ then $x^2 \in P$. In particular $1 \in P$ in an ordered field.

Example 68. *The field of rational numbers* $\mathbb{Q}$ *together with the set*

$$P = \left\{ \frac{a}{b} \in \mathbb{Q}\,;\, a \cdot b \in \mathbb{N} \right\}$$

is an ordered field.

Example 69. *The field* $\mathbb{Z}_p$ *can't be ordered, since if we add* $\bar{1}$, *p times, the result is* $\bar{0}$, *i.e.* $\bar{1} + \cdot + \bar{1} = \bar{0}$, *but in an ordered field the sum of positive elements has to be positive, in particular nonzero.*

Example 70. *The field* $\mathbb{Q}(t)$ *of example 66 together with the set*

$$P = \left\{ \frac{p(t)}{q(t)}; \text{ the leading coefficient of } p(t) \cdot q(t) \text{ is positive} \right\}$$

is an ordered field.

In an ordered field K, if $x - y \in P$ we write $x > y$ (or $y < x$). In particular, $x > 0$ implies $x \in P$ and $x < 0$ implies $x \in -P$. Notice that if $x \in P$ and $y \in -P$ then $x > y$.

We use the notation $x \leq y$ to indicate $x < y$ or $x = y$, in a similar way we can define $x \geq y$ as well.

Proposition 71. *Let* K *be an ordered field and* $x, y, z \in K$*, then*

(I) (Transitivity) $x < y$ *and* $y < z \Rightarrow x < z$*;*

(II) (Trichotomy) Only one of the following occurs: $x = y$*,* $x > y$*,*$x < y$*;*

(III) (Sum monotoneity) $x < y \Rightarrow x + z < y + z$*;*

(IV) (Multiplication monotoneity)If $z > 0$*, then* $x < y \Rightarrow x \cdot z < y \cdot z$ *and if* $z < 0$*, then* $x < y \Rightarrow x \cdot z > y \cdot z$*.*

Since in an ordered field K, 1 is always positive we have $1 + 1 > 1 > 0$ and $1 + 1 + 1 > 1 + 1$, so we can easily define an increasing injection

$$f : \mathbb{N} \to K$$

by $f(n) = \overbrace{1 + 1 + \cdots + 1}^{n}$, or more precisely, $f(1) = 1$ and $f(n+1) = f(n)+1$. Therefore, it makes sense to identify $\mathbb{N}$ with $f(\mathbb{N}) \subseteq K$, so henceforward we will simply write

$$\mathbb{N} \subseteq K$$

whenever K is an ordered field.

Notice in particular that $f(n)$ is never zero in this case, hence every ordered field is infinite. Whenever $f(n)$ is never zero, for f defined above, we say K has **characteristic zero**; if $f(p) = 0$, then we say K has **characteristic p**.

Example 72. *The field* $\mathbb{Q}$ *clearly has characteristic zero. The field* $\mathbb{Z}_p$ *has characteristic* p*.*

Proceeding as before, we can extend the bijection above to $f : \mathbb{Z} \to K$ and view $\mathbb{Z} \subseteq K$ as well. Hence, we have $\mathbb{N} \subseteq \mathbb{Z} \subseteq K$.

Finally, we can use $f : \mathbb{Z} \to K$ to define a bijection $g : \mathbb{Q} \to K$ by $g(\frac{a}{b}) = f(a) \cdot f(b)^{-1}$. So we may identify $\mathbb{Q}$ with $g(\mathbb{Q}) \subseteq K$ and write

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq K$$

whenever K is an ordered field.

Example 73. *If $K = \mathbb{Q}$ in the above discussion, then $g : \mathbb{Q} \to \mathbb{Q}$ is the identity automorphism. i.e. $g(\frac{a}{b}) = \frac{a}{b}$.*

Proposition 74. *(Bernoulli's inequality) Let K be an ordered field and $x \in K$. If $x \geq -1$ and $n \in \mathbb{N}$, then*

$$(1 + x)^n \geq 1 + n \cdot x$$

Proof. We use induction on $n \in \mathbb{N}$. The case $n = 1$ is clear, suppose the result valid for n. Then $(1 + x)^{n+1} = (1 + x)^n(1 + x) \geq (1 + n \cdot x)(1 + x) = 1 + x + n \cdot x + x^2 \geq 1 + x + n \cdot x$, as expected. *(Notice that we used the fact that $x \geq -1$ in the first inequality and proposition 71(IV).)* □

11 Intervals

Let K be an ordered field and $a < b$ be elements of K. We call any subset of the following form an interval:

$[a, b] = \{x \in K; a \leq x \leq b\}$ *(closed interval)*

$(a, b) = \{x \in K; a < x < b\}$ *(open interval)*

$[a, b) = \{x \in K; a \leq x < b\}$ and $(a, b] = \{x \in K; a < x \leq b\}$

$(-\infty, b) = \{x \in K; x < b\}$ and $(-\infty, b] = \{x \in K; x \leq b\}$

$(a, \infty) = \{x \in K; a < x\}$ and $[a, \infty) = \{x \in K; a \leq x\}$

$(-\infty, \infty) = K$

If $a = b$, then $[a, a] = a$ and $(a, a) = \emptyset$. We say the interval $[a, a]$ is degenerate.

Let K be an ordered field and $x \in K$. We define the absolute value of x, denoted by $|x|$, by

$$|x| := \max\{x, -x\},$$

which is to say, $|x|$ is the greater of the two numbers x or $-x$. Geometrically, if the elements of K are put in a straight line, $|x|$ measures the distance between x and 0, hence $|x - a|$ is the distance between x and a.

Theorem 75. *Let x, y be elements of an ordered field K. The following are equivalent:*

(i) $-y \leq x \leq y$

(ii) $x \leq y$ *and* $-x \leq y$

(iii) $|x| \leq y$

Corollary 76. *Let $x, a, \epsilon \in K$ then*

$$|x - a| \leq \epsilon \iff a - \epsilon \leq x \leq a + \epsilon.$$

Remark 2. *The theorem and corollary remains valid if we exchange $\leq$ by $<$.*

Theorem 77. *Let x, y, z be elements of an ordered field K.*

(i) $|x + y| \leq |x| + |y|$;

(ii) $|x \cdot y| = |x| \cdot |y|$;

(iii) $|x| - |y| \leq ||x| - |y|| \leq |x - y|$;

(iv) $|x - z| \leq |x - y| + |y - z|$.

Let K be an ordered field and $X \subseteq K$. An **upper bound** of X is an element $M \in K$ such that $x \leq M$ for every $x \in X$. Similarly, a **lower bound** is an element $m \in K$ such that $m \leq x$ for every $x \in X$. We say X is *bounded from above* if it has an upper bound, *bounded from below* if it has a lower bound, and *bounded* if it has upper and lower bounds, i.e. $X \subseteq [m, M]$.

Example 78. *The principle of well-ordering guarantees that $\mathbb{N}$ is bounded from below when viewed as a set inside the ordered field $\mathbb{Q}$. $\mathbb{N}$ is obviously not bounded from above in $\mathbb{Q}$, since given any n, $n + 1 > n$.*

Example 79. *Oddly enough, $\mathbb{N}$ is bounded from above in the ordered field $\mathbb{Q}(t)$ from example 70. Since given any $n \in \mathbb{N}$, the rational function $r(t) = t$ satisfies $r(t) - n > 0$. Therefore, $r(t) \in \mathbb{Q}(t)$ is an upper bound for $\mathbb{N}$ and the latter is bounded from above, hence bounded, in $\mathbb{Q}(t)$.*

Theorem 80. *Let K be an ordered field. The following are equivalent:*

1. *$\mathbb{N}$ is not bounded from above;*
2. *Given $a, b \in K$, with $a > 0$, $\exists n \in \mathbb{N}$ such that $n \cdot a > b$;*
3. *Given $a > 0$ in K, $\exists n \in \mathbb{N}$ such that $0 < \frac{1}{n} < a$.*

*A field K satisfying the above conditions is called **Archimedean field**.*

Proof. The proof is based on the implications $1 \Rightarrow 2$, $2 \Rightarrow 3$, $3 \Rightarrow 1$.

($1 \Rightarrow 2$) Since $\mathbb{N}$ is unbounded, $\frac{b}{a} < n$ for some $n \in \mathbb{N}$, hence $n \cdot a > b$.

($2 \Rightarrow 3$) Take $b = 1$ in 2.

($3 \Rightarrow 1$) For any $a > 0$, consider $\frac{1}{a}$, by 3., $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{1}{a} \iff n > a$. Therefore, no positive element is an upper bound. Similarly, no negative element can be an upper bound since if x is negative $-x$ is positive and we can apply the same argument.

□

Example 81. *Examples 78 and 79 say that $\mathbb{Q}$ is Archimedean but $\mathbb{Q}(t)$ isn't.*

12 The real numbers $\mathbb{R}$

Let K be an ordered field and $X \subseteq K$ be a bounded from above subset. The **supremum** of X, denoted $\sup X$ is the least upper bound of X, in other words, among all upper bounds $M \in K$ of X, i.e. $x \leq M$ for every $x \in X$, $\sup X \in K$ is the least of them. Therefore, $\sup X \in K$ has the following properties:

(i) (upper bound) For every $x \in X$, $x \leq \sup X$.

(ii) (least upper bound) Given any $a \in K$ such that $x \leq a$ for every $x \in X$, then $\sup X \leq a$. In other words, if $a < \sup X$ then $\exists b \in X$ such that $a < b$.

Lemma 82. *If the supremum of a set X exists, it is unique.*

Proof. Suppos $a = \sup X$ and $b = \sup X$. By (ii) above, $a \leq b$ since a is the least upper bound, but for the same reason we also have $b \leq a$, hence $a = b$. $\square$

Lemma 83. *If a set X has a maximum element, then $\max X = \sup X$.*

Proof. Indeed, $\max X$ is obviously an upper bound and any other upper bound is greater than or equal to the maximum. $\square$

Example 84. *Consider the set $I_n = \{1, 2, \ldots, n\} \subseteq \mathbb{Q}$. Then $\sup I_n = \max I_n = n$.*

Example 85. *Consider the set $X = \{-\frac{1}{n}; n \in \mathbb{N}\} \subseteq \mathbb{Q}$. Then $\sup X = 0$. Indeed, 0 is an upper bound and given any number $a < 0$ we can find $-\frac{1}{n}$ such that $a < -\frac{1}{n}$ since $\mathbb{Q}$ is an Archimedean field.*

Similar to the idea of supremum, the **infimum** of a bounded from below set $X \subseteq K$, denoted $\inf X$, is the greatest lower bound. The element $\inf X \in K$ has the following properties:

(i) (lower bound) For every $x \in X$, $x \geq \inf X$.

(ii) (greatest lower bound) Given any $a \in K$ such that $x \geq a$ for every $x \in X$, then $\inf X \geq a$.

The lemmas 82 and 83 extend naturally to the notion of infimum, namely, if $X \subseteq K$ has a minimum element m then $m = \inf X$. Additionally, the infimum is unique. More generally, we easily conclude that:

Proposition 86. *Let $X \subseteq K$ be a bounded subset of an ordered field K. Then, $\inf X \in X \iff \inf X = \min X$ and $\sup X \in X \iff \sup X = \max X$. In particular, every finite set has a supremum and infimum.*

Example 87. *Consider the set $X = (a, b)$, an open interval in a ordered field K. Then $\inf X = a$ and $\sup X = b$. Indeed, a is a lower bound, by definition of interval, suppose $c > a$, we claim c can't be a lower bound. For instance, consider $d = \frac{a+c}{2} \in (a, b)$. We have $d < c$ if $c < b$, hence the conclusion.*

Example 88. *Let* $X = \{\frac{1}{2^n}; n \in \mathbb{N}\} \subseteq \mathbb{Q}$. *Then* $\inf X = 0$ *and* $\sup X = \frac{1}{2}$. *Notice that* $\max X = \frac{1}{2}$, *by lemma 83* $\sup X = \frac{1}{2}$. *Now,* 0 *is obviously a lower bound. Suppose* $c > 0$, *since* $\mathbb{Q}$ *is Archimedean we can find* $n \in \mathbb{N}$ *such that* $n + 1 > \frac{1}{c}$. *By Bernoulli's inequality (Proposition 74), we have* $2^n = (1 + 1)^n \geq 1 + n > \frac{1}{c}$, *hence* $c > \frac{1}{2^n}$ *and* c *can't be a lower bound, so* $\inf X = 0$.

Lemma 89. *(Pythagoras) There is no* $x \in \mathbb{Q}$ *satisfying* $x^2 = 2$.

Proof. Suppose not, then $x = \frac{p}{q}$ satisfies $\left(\frac{p}{q}\right)^2 = 2$, or $p^2 = 2q^2$, where $p, q \in \mathbb{Z}$ and $q \neq 0$. If we decompose p^2 in prime factors, it will have an even number of factors equal to two, the same occurs for q^2. Since $2q^2$ has an odd number of factors two, we can't have $p^2 = 2q^2$. □

Proposition 90. *Consider the sets of rational numbers* $X = \{x \in \mathbb{Q}; x \geq 0$ *and* $x^2 < 2\}$ *and* $Y = \{y \in \mathbb{Q}; y > 0$ *and* $y^2 > 2\}$. *There are no rational numbers* $a, b \in \mathbb{Q}$ *such that* $a = \sup X$ *and* $b = \inf Y$.

Proof. We prove the result concerning the supremum, the result about infimum can be proven similarly. We first claim X doesn't have a maximum element. Given $x \in X$, take $r < 1$ satisfying $0 < r < \frac{2-x^2}{2x+1}$, then $x + r \in X$, so $x \in X$ can't be the maximum. Indeed, since $r < 1 \Rightarrow r^2 < r$, and we have

$$(x + r)^2 = x^2 + 2xr + r^2 < x^2 + 2xr + r = x^2 + r(2x + 1) < x^2 + 2 - x^2 = 2.$$

By a similar reasoning, given $y \in Y$, it's possible to find $r > 0$ such that $y - r \in Y$, so Y doesn't have a minimum element. Finally, notice that if $x \in X$, $y \in Y$ then $x < y$, since $x^2 < 2 < y^2 \Rightarrow 0 < (x - y)(x + y) \Rightarrow 0 < (x - y)$.

Suppose there is a number $a \in \mathbb{Q}$ such that $a = \sup X$. Then $a \notin X$, otherwise it would be its maximum. If $a \in Y$, since Y doesn't have a minimum, there would be a $b \in Y$ such that $b < a$, then $x < b < a$, a contradiction since a is the supremum. We conclude that $a \notin X$ and $a \notin Y$, so a has to satisfy $a^2 = 2$, a contradiction by lemma 89. □

Since every ordered field contains $\mathbb{Q}$, in the proposition above, if there is an ordered field K such that every nonempty bounded from above set has a supremum, then $a = \sup X$ is an element of K satisfying $a^2 = 2$.

Example 91. *(A bounded set with no supremum) Let* K *be a non-Archimedean field. Then, by definition,* $\mathbb{N} \subseteq K$ *is bounded from above. Let* $M \in K$ *be an* *upper bound for* $\mathbb{N}$*. So* $n+1 \leq M$ *for all* $n \in \mathbb{N}$*, but then* $n \leq M-1$ *and* $M-1$ *is also an upper bound. We conclude that if* M *is an upper bound,* $M-1$ *is one as well, hence* $\sup \mathbb{N}$ *doesn't exists in* K*.*

We say that an ordered field K is **complete**, if every nonempty bounded from above subset $X \subseteq K$ has a supremum in K. This motivates the following axiom (also called **the fundamental axiom of mathematical analysis**):

Axiom. There is a complete ordered field, represented by $\mathbb{R}$, called the field of real numbers.

Remark 3. *Notice that in a complete ordered field* K*, if* $X \subseteq K$ *is bounded from below then* X *has an infimum.*

Remark 4. *From example 91 we conclude that every complete ordered field is Archimedean.*

Proposition 92. *If* K, L *are complete ordered fields, then there is an isomorphism* $f : K \to L$*.*

The proposition above says that, in some suitable sense, $\mathbb{R}$ is the only complete ordered field.

Until the end of the semester, **every** topic we discuss will involve the complete ordered field $\mathbb{R}$ and its properties.

The discussion above leads to the conclusion that despite there is no number $x \in \mathbb{Q}$ satisfying $x^2 = 2$, there is a positive number $x \in \mathbb{R}$ such that $x^2 = 2$. We denote that number by $\sqrt{2}$. There is nothing special about 2, so we can generalize the proof above to any $n \in \mathbb{N}$ that is not a perfect square and conclude that we can find a positive number, denoted by $\sqrt{n}$, such that $(\sqrt{n})^2 = n$.

We can generalize even further and talk about the n^{th}*-root* of $m \in \mathbb{N}$, denote by $\sqrt[n]{m}$. Namely, a positive number $x \in \mathbb{R}$ such that $x^n = m$.

We call the elements of the set $\mathbb{R} - \mathbb{Q}$, **irrational numbers**. As we've just seen, there are many of them, namely, numbers of the form $\sqrt[n]{2}$, for $n \geq 2$, are all irrational. In fact, we shall see next that irrational numbers are everywhere, in a precise sense, as a subset of the real numbers.

A subset $X \subseteq \mathbb{R}$ is said to be **dense** in $\mathbb{R}$ if for every $a, b \in \mathbb{R}$, with $a < b$, we can find $x \in X$ such that $a < x < b$. In other words, X is dense in $\mathbb{R}$ if every open non-degenerate interval (a, b) contains a point $x \in X$.

Example 93. *Let $X = \mathbb{R} - \mathbb{Z}$. Then X is dense in $\mathbb{R}$. Indeed, every open interval (a, b) is an infinite set (since $\mathbb{R}$ is ordered). On the other hand, $\mathbb{Z} \cap (a, b)$ is finite, hence we can always find a number $x \notin \mathbb{Z}$ with $x \in (a, b)$.*

Theorem 94. *The set of rational numbers, $\mathbb{Q}$, and the set of irrational numbers, $\mathbb{R} - \mathbb{Q}$, are both dense in $\mathbb{R}$.*

Proof. Let $(a, b) \in \mathbb{R}$ be a non-degenerate open interval. The idea of the proof is that since $b - a > 0$, there is a natural number $n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$, hence a multiple of this number, say $\frac{m}{n}$ eventually will be in (a, b). More formally, let $X = \{m \in \mathbb{Z}; \frac{m}{n} \geq b\}$. Since $\mathbb{R}$ is Archimedean, $X \neq \emptyset$. Notice that X is bounded from below by $nb \in \mathbb{R}$. By the well ordering principle, X has a smallest element, say $m_0 \in X$. By the smallness of m_0, the number $m_0 - 1 \notin X$, so $\frac{m_0 - 1}{n} < b$. We claim $a < \frac{m_0 - 1}{n}$. Suppose not, then $\frac{m_0 - 1}{n} \leq a < b < \frac{m_0}{n}$, which implies that $b - a \leq \frac{m_0}{n} - \frac{m_0 - 1}{n} = \frac{1}{n}$, a contradiction. Therefore, the rational number $\frac{m_0 - 1}{n}$ satisfies $a < \frac{m_0 - 1}{n} < b$ and $\mathbb{Q}$ is dense in $\mathbb{R}$. We can apply the same argument *mutatis mutandis* to conclude that $\mathbb{R} - \mathbb{Q}$ is dense. Namely, instead of using $\frac{1}{n}$ in our argument, we use an irrational number, say $\frac{\sqrt{2}}{n}$. □

Theorem 95. *(The nested intervals principle) Let $I_1 \supseteq I_2 \supseteq \ldots I_n \supseteq \ldots$ be a decreasing sequence of closed intervals of the form $I_n = [a_n, b_n]$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$, or more precisely,*

$$\bigcap_{n=1}^{\infty} I_n = [a, b],$$

where $a = \sup a_n = \sup\{a_n; n \in \mathbb{N}\}$ and $b = \inf b_n = \inf\{b_n; n \in \mathbb{N}\}$

Proof. By hypothesis, $I_n \supseteq I_{n+1}, \forall n \in \mathbb{N}$, which implies:

$$a_1 \leq a_2 \leq \ldots a_n \leq \ldots \leq b_n \leq \ldots \leq b_2 \leq b_1.$$

Notice that a_n is bounded from above by b_1, hence the supremum of a_n, $a \in \mathbb{R}$, is well defined. Similarly, the infimum of b_n, $b \in \mathbb{R}$, is well defined. Since b_n is an upper bound for a_n, we have $a \leq b_n, \forall n \in \mathbb{N}$. On the other hand, a is also an upper bound and we conclude that

$$a_n \leq a \leq b_n, \forall n \in \mathbb{N}.$$

A similar reasoning can be applied to b, hence

$$[a, b] \subseteq I_n, \forall n \in \mathbb{N}.$$

If $x < a$, we can find a_{n_0} such that $x < a_{n_0}$, so $x \notin I_{n_0} \Rightarrow x \notin \bigcap_{n=1}^{\infty} I_n$. Similarly, if $x > b$, then we can find n_1 such that $b_{n_1} < x$, so $x \notin I_{n_1} \Rightarrow x \notin \bigcap_{n=1}^{\infty} I_n$. We conclude that $\bigcap_{n=1}^{\infty} I_n = [a, b]$. □

Theorem 96. $\mathbb{R}$ *is uncountable.*

Proof. Let $X = \{x_1, x_2, \ldots\} \subseteq \mathbb{R}$ be a countable subset of $\mathbb{R}$, which we know exists by theorem 48. We claim there is always an $x \in \mathbb{R}$ such that $x \notin X$. Pick a closed interval I_1 not containing x_1, this is possible since $\mathbb{R}$ is infinite. Proceed by induction, after setting I_n not containing x_n, we select $I_{n+1} \subseteq I_n$ as a closed interval which doesn't contain x_{n+1}. Proceeding this way, we construct a nested sequence of closed intervals $I_1 \supseteq I_2 \supseteq \ldots I_n \supseteq \ldots$. Therefore, by theorem 95, there is at least one $x \in \mathbb{R}$ that is not in X. □

Corollary 97. *Any non-degenerate interval* $(a, b) \subseteq \mathbb{R}$ *is uncountable.*

Proof. The function $f : (0, 1) \to (a, b)$ defined by $f(x) = (b-a)x+a$ is bijective, so it suffices to prove the result for $(0, 1)$. Suppose $(0, 1)$ is countable, then $(0, 1]$ is also countable and reasoning as before, $(n, n + 1]$ is countable for every $n \in \mathbb{N}$. Then $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (n, n + 1]$ is countable, a contradiction. □

Corollary 98. *The set of irrational numbers* $\mathbb{R} - \mathbb{Q}$ *is uncountable.*

Proof. Suppose not, then $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} - \mathbb{Q})$ is countable, a contradiction. □

13 Sequences

A **sequence of real numbers**, denoted by $x_n := x(n)$, is a function $x : \mathbb{N} \to \mathbb{R}$ that associates to each natural number $n \in \mathbb{N}$, a real number $x(n) \in \mathbb{R}$. There is no universally defined notation for a sequence x_n, but here are examples of common notation found in the literature:

$$\{x_n\}_{n \in \mathbb{N}},\ \mathbf{x}_n,\ \{x_1, x_2, \ldots\},\ (x_n)$$

We say that a sequence x_n is *bounded* if there are $a, b \in \mathbb{R}$ such that

$$a \leq x_n \leq b,$$

this is equivalent of saying that $x(\mathbb{N}) \subseteq [a, b]$, i.e. $x(n)$ is bounded as a function. A sequence is *unbounded* when is not bounded.

A sequence x_n is *bounded from above* when there is $b \in \mathbb{R}$ such that $x_n \leq b$, and *bounded from below* if there is an $a \in \mathbb{R}$ such that $a \leq x_n$. Notice that a sequence is bounded if and only if is bounded from above and below.

Let $K \subseteq \mathbb{N}$ be an infinite subset. Then K is countably infinite, let $b : \mathbb{N} \to K$, given by $k \mapsto n_k$ be a bijection. Given any sequence $x : \mathbb{N} \to \mathbb{R}$, the composition $x_{n_k} := x \circ b : K \to \mathbb{R}$ is also a sequence, called a **subsequence** of x_n.

Example 99. *Let $K = \{n; n \text{ is even}\} \subseteq \mathbb{N}$ and $b(k) = 2k$. In this case, given a sequence x_n, the sequence $x_{n_k} := x_{2n}$ is a subsequence of x_n. For example, if $x_n = (-1)^n$,i.e. $\{-1, 1, -1, \ldots\}$, then x_{2n} is the constant subsequence $x_{2n} = \{1, 1, 1, \ldots\}$.*

Notice that every subsequence x_{n_k} of a bounded sequence x_n is itself bounded by definition. We say a sequence x_n is *nondecreasing* if $x_n \leq x_{n+1}, \forall n \in \mathbb{N}$, and if the inequality is strict, i.e. $x_n < x_{n+1}$, we call x_n an *increasing* sequence. We define *nonincreasing* and *decreasing* sequences in a similar way by placing $\geq$ ($>$) instead of $\leq$ ($<$).

A sequence that is either nondecreasing, nonincreasing, increasing, or decreasing will be called **monotone**.

Lemma 100. *A monotone sequence x_n is bounded $\iff$ it has a bounded subsequence.*

Proof. Only the converse is not obvious. Suppose x_{n_k} is a bounded monotone subsequence, say $x_{n_1} \leq x_{n_2} \leq \ldots \leq b$. Given any $n \in \mathbb{N}$, we can find $n_k > n$, hence $x_n \leq x_{n_k} \leq b$. □

Example 101. $x_n = 1$*, i.e.* $\{1, 1, 1, \ldots\}$*, is a constant, bounded, nonincreasing and nondecreasing sequence.*

Example 102. $x_n = n$*, i.e.* $\{1, 2, 3, \ldots\}$*, is an unbounded increasing sequence.*

Example 103. $x_n = \frac{1}{n}$, *i.e.* $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, *is a bounded decreasing sequence, since* $0 < x_n \leq 1$.

Example 104. $x_n = 1 + (-1)^n$, *i.e.* $\{0, 2, 0, 2, \ldots\}$, *is a bounded sequence that is not monotone.*

Example 105. $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$ *is increasing, and bounded, since* $0 < x_n < 1 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{n-1}} < 3$. *The sequence* $y_n = (1 + \frac{1}{n})^n$ *is related to this sequence, since by the binomial theorem* $y_n \leq x_n$, *therefore* y_n *is also bounded,* $0 < y_n < 3$.

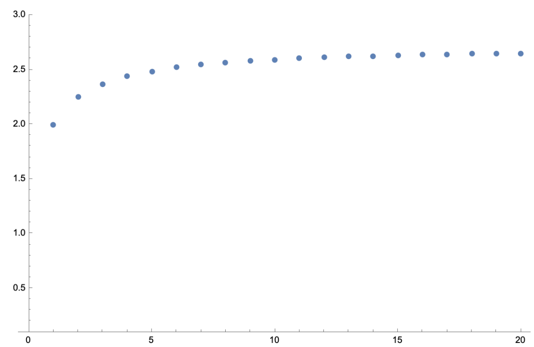


Figure 1: $y_n = (1 + \frac{1}{n})^n$

Example 106. *Let* $x_1 = 0$ *and* $x_2 = 1$, *and consider, by induction,* $x_{n+2} = x_{n+1} + x_n$. *It's easy to see that* $0 \leq x_n \leq 1$, *and moreover a quick computation shows that* $x_{2n} = 1 - \left(\frac{1}{4} + \frac{1}{16} + \ldots + \frac{1}{4^{n-1}}\right)$ *and* $x_{2n+1} = \frac{1}{2}\left(1 + \frac{1}{4} + \frac{1}{16} + \ldots + \frac{1}{4^{n-1}}\right)$. *So* x_n *is a bounded sequence that is not monotone.*

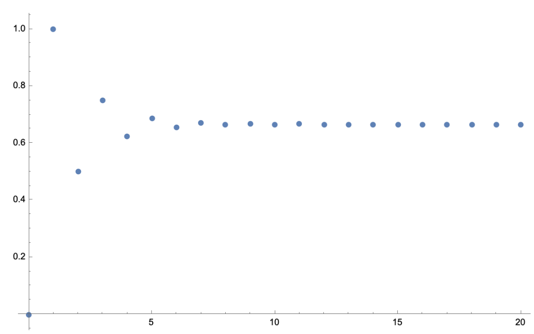


Figure 2: $x_{n+2} = x_{n+1} + x_n$

Example 107. *Let $a \in \mathbb{R}$ such that $0 < a < 1$. The sequence $x_n = 1 + a + a^2 + \ldots + a^n = \frac{1-a^{n+1}}{1-a}$ is increasing, since $a > 0$, and bounded since $0 < x_n \leq \frac{1}{1-a}$.*

Example 108. *The sequence $\{1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \ldots\}$ given by $x_n = \sqrt[n]{n}$, increases for $n = 1, 2$. We claim that starting at the third term, this sequence is decreasing. Indeed, $x_{n+1} < x_n$ is equivalent to $(n+1)^n < n^{n+1}$, which is equivalent to $(1 + \frac{1}{n})^n < n$, which is true for $n \geq 3$ by Example 105. Hence, x_n is bounded.*

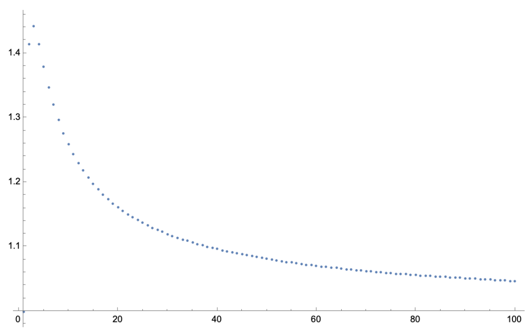


Figure 3: $x_n = \sqrt[n]{n}$

14 The limit of a sequence

Informally, to say $a \in \mathbb{R}$ is the limit of the sequence x_n is to say that the terms of the sequence are very close to a, when n is large. More precisely, we quantify this using the following definition:

$$\lim_{n \to \infty} x_n = a \coloneqq \forall \epsilon > 0 \, \exists n_0 \in \mathbb{N}; n > n_0 \Rightarrow |x_n - a| < \epsilon$$

In other words: *"The limit of sequence x_n is a, if for every positive number ϵ, no matter how small it is, it's always possible to find an index n_0 such that the distance between x_n and a is less then ϵ, for $n > n_0$."*

Additionally, the above is the same of saying that any open interval

$$(a - \epsilon, a + \epsilon)$$

centered at a and with length 2ϵ, contains all the points of the sequence x_n except possibly a finite amount of them.

Remark 5. *It's a common practice to omit "$n \to \infty$ "and write* $\lim x_n$ *only.*

When $\lim x_n = a$, we say x_n *converges* to a, also denoted by $x_n \to a$, and call x_n *convergent*. If x_n is not convergent, we call it *divergent*, i.e. there is no $a \in \mathbb{R}$ such that $\lim x_n = a$.

Theorem 109. *(Uniqueness of the limit) If* $\lim x_n = a$ *and* $\lim x_n = b$*, then* $a = b$*.*

Proof. Let $\lim x_n = a$ and $b \neq a$, it's enough to prove that $\lim x_n \neq b$. Take $\epsilon = \frac{|b-a|}{2}$, then since $\lim x_n = a$, we can find n_0 such that $n > n_0 \Rightarrow |x_n - a| < \epsilon$. Therefore, $x_n \notin (b - \epsilon, b + \epsilon)$ if $n > n_0$ and we can't have $\lim x_n = b$. □

Theorem 110. *If* $\lim x_n = a$*, then for every subsequence* x_{n_k} *of* x_n*, we also have* $\lim x_{n_k} = a$*.*

Proof. Indeed, since given $\epsilon > 0$ it's possible to find n_0 such that $n > n_0 \Rightarrow |x_n - a| < \epsilon$, the same n_0 works for x_{n_k} as well, namely, $n_k > n_0 \Rightarrow |x_{n_k} - a| < \epsilon$. □

Corollary 111. *Let* $k \in \mathbb{N}$*. If* $\lim x_n = a$ *then* $\lim x_{n+k} = a$*, since* x_{n+k} *is a subsequence of* x_n*.*

In other words, Corollary 111 says that the limit of a sequence doesn't change if we omit the first k terms.

Theorem 112. *Every convergent sequence* x_n *is bounded.*

Proof. Suppose $\lim x_n = a$. Then it's possible to find n_0 such that $x_n \in (a-1, a+1)$ for $n > n_0$. Let $M = \max\{|x_1|, \ldots, |x_{n_0}|, |a-1|, |a+1|\}$, then $x_n \in (-M, M)$. □

Example 113. *The sequence* $\{0, 1, 0, 1, 0, 1, \ldots\}$ *can't be convergent by theorem 110, since it has two subsequences converging to different values, namely,* $x_{2n} = 1$ *and* $x_{2n-1} = 0$*. Also, this sequence is an example of a bounded sequence which is not convergent, illustrating the fact that the converse of theorem 112 is false.*

Theorem 114. *Every bounded monotone sequence is convergent.*

Proof. Suppose $x_n \leq x_{n+1}$, the other cases are proved similarly. Since x_n is bounded, $\sup x_n$ is well defined, say $a = \sup x_n$. Let $\epsilon > 0$ be given, then $\exists n_0 \in \mathbb{N}$ such that $a - \epsilon < x_{n_0}$, but since $x_n \leq x_{n+1}$, we must have have $a - \epsilon < x_n, \forall n \geq n_0$. We obviously have $x_n \leq a$, hence $a - \epsilon < x_n < a + \epsilon$ for $n > n_0$ and $\lim x_n = a$. □

Corollary 115. *If a monotone sequence x_n has a convergent subsequence then x_n is convergent.*

Example 116. *Every constant sequence $x_n = k \in \mathbb{R}$ is convergent and $\lim x_n = k$.*

Example 117. *The sequence $\{1, 2, 3, 4, \ldots\}$ is divergent because it's unbounded.*

Example 118. *The sequence $\{1, -1, 1, -1, \ldots\}$ is divergent because it has two subsequences converging to different values.*

Example 119. *The sequence $x_n = \frac{1}{n}$ is convergent and $\lim x_n = 0$, since $\mathbb{R}$ is Archimedian and given any $\epsilon > 0$ it's possible to find $n_0 \in \mathbb{N}$ such that $0 < \frac{1}{n_0} < \epsilon$. Hence, $n > n_0 \Rightarrow \frac{1}{n} < \epsilon$.*

Example 120. *Let $0 < a < 1$. The sequence $x_n = a^n$ is monotone and bounded, hence convergent. Notice that $\lim x_n = 0$ in this case.*

15 Properties of limits

Theorem 121. *Let $\lim x_n = 0$ and y_n a bounded sequence. Then*

$$\lim x_n \cdot y_n = 0.$$

Proof. Let $c > 0$ be such that $|y_n| < c$. Let $\epsilon > 0$ be given, and $n_0 \in \mathbb{N}$ a number such that $n > n_0 \Rightarrow |x_n| < \frac{\epsilon}{c}$. Then, $n > n_0 \Rightarrow |x_n y_n| < \frac{\epsilon}{c} \cdot c = \epsilon$. □

Example 122. *Using the theorem above we have $\lim_{n \to \infty} \frac{\sin n}{n} = 0$*

Theorem 123. *Let $\lim x_n = a$ and $\lim y_n = b$. Then*

1. $\lim x_n + y_n = a + b$, $\lim x_n - y_n = a - b$;
2. $\lim x_n \cdot y_n = ab$;

3. If $b \neq 0$ *then* $\lim \frac{x_n}{y_n} = \frac{a}{b}$

Example 124. *Let* $a \in \mathbb{R}$ *be a positive number. The sequence* $x_n = \sqrt[n]{a}$ *is bounded and monotone, hence converges. We claim*

$$\lim \sqrt[n]{a} = 1.$$

Indeed, let $L := \lim \sqrt[n]{a}$ *and consider the subsequence* $y_n = x_{n(n+1)}$ *then*

$$L = \lim y_n = \lim a^{\frac{1}{n(n+1)}} = \lim a^{\frac{1}{n} - \frac{1}{n+1}} = \frac{\lim a^{\frac{1}{n}}}{\lim a^{\frac{1}{n+1}}} = 1$$

Example 125. *Similar to the example above is the sequence* $x_n = \sqrt[n]{n}$*. It is bounded and monotone (starting from the third term), hence converges. We claim*

$$\lim \sqrt[n]{n} = 1.$$

Let $L := \lim \sqrt[n]{n}$ *and consider the subsequence* $y_n = x_{2n}$ *then*

$$L^2 = \lim y_n \cdot y_n = \lim \sqrt[n]{2n} = \lim \sqrt[n]{2}\sqrt[n]{n} = 1 \cdot L = L$$

Hence, $L = 0$ *or* $L = 1$*, but* $L \neq 0$ *since* $x_n \geq 1$*.*

Theorem 126. *If* $\lim x_n = a$ *and* $a > 0$*, then* $\exists n_0$ *such that* $x_n > 0$ *for* $n > n_0$*. An equivalent statement is valid if* $a < 0$*, namely, up to a finite amount of indexes,* $x_n < 0$*.*

Proof. It's possible to find n_0 such that $n > n_0 \Rightarrow |x_n - a| < \frac{a}{2}$, in particular, $x > \frac{a}{2} > 0$ if $n > n_0$. The case $a < 0$ is proved similarly. □

Corollary 127. *If* x_n, y_n *are convergent sequences and* $x_n \leq y_n$ *then* $\lim x_n \leq \lim y_n$*.*

Corollary 128. *If* x_n *is convergent and* $x_n \geq a \in \mathbb{R}$ *then* $\lim x_n \geq a$*.*

Theorem 129. *(Squeeze theorem) If* $x_n \leq y_n \leq z_n$ *and* $\lim x_n = \lim z_n$*, then* $\lim y_n = \lim x_n = \lim z_n$*.*

16 $\liminf x_n$ and $\limsup x_n$

A number $a \in \mathbb{R}$ is an accumulation point of the sequence x_n, if there is a subsequence x_{n_k} such that $\lim_{k\to\infty} x_{n_k} = a$.

Theorem 130. *$a \in \mathbb{R}$ is an accumulation point of the sequence x_n if and only if $\forall \epsilon > 0$, there are infinitely many values of $n \in \mathbb{N}$ such that $x_n \in (a-\epsilon, a+\epsilon)$.*

Proof. The implication is clear, we prove the converse only. Take $\epsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{k}, \ldots$, then it's possible to find x_{n_k} such that $|x_{n_k} - a| < \frac{1}{k}$ for every $k \in \mathbb{N}$ and moreover $n_k < n_{k+1}$, in particular, $\lim_{k\to\infty} x_{n_k} = a$. $\square$

Example 131. *If $\lim x_n = a$ then x_n has only one accumulation point, namely $a \in \mathbb{R}$. This follows directly from theorem 110.*

Example 132. *The sequence $\{0, 1, 0, 2, 0, 3, \ldots\}$ is divergent. However, it has 0 as an accumulation point, due to the constant subsequence $x_{2n-1} = 0$. Similarly, the divergent sequence $\{1, -1, 1, -1, 1, -1, \ldots\}$ has only two accumulation points: 0 and 1. The divergent sequence $\{1, 2, 3, 4, 5, 6, \ldots\}$ doesn't have an accumulation point.*

Example 133. *By theorem 94, every real number $r \in \mathbb{R}$ is an accumulation point of a sequence of rational numbers.*

We shall see below that every bounded sequence has at least two accumulation points, and the sequence converges if and only if they coincide.

Let x_n be a bounded sequence, say $m \le x_n \le M$, with $m, M \in \mathbb{R}$. Set

$$X_n = \{x_n, x_{n+1}, \ldots\}.$$

Then $X_n \subseteq [m, M]$ and $X_{n+1} \subseteq X_n$. Define $a_n := \inf X_n$ and $b_n := \sup X_n$, then

$$m \le a_1 \le a_2 \le \ldots \le a_n \le \ldots \le b_n \le \ldots \le b_2 \le b_1 \le M,$$

and the following limits are well defined $a = \lim a_n = \sup a_n$ and $b = \lim b_n = \inf b_n$. We define the *limit inferior* of x_n as

$$\liminf x_n := a$$

and the *limit superior* of x_n as

$$\limsup x_n := b.$$

We obviously have

$$\liminf x_n \le \limsup x_n.$$

Example 134. *Consider the sequence* $x_n = \{0, 1, 0, 1, 0, 1, \ldots\}$*. Using the notation above,* $a_n \equiv 0$ *and* $b_n \equiv 1$*. Therefore,* $\liminf x_n = 0$ *and* $\limsup x_n = 1$*. More generally, we have:*

Theorem 135. *Let* x_n *be a bounded sequence. Then* $\liminf x_n$ *is the smallest accumulation point and* $\limsup x_n$ *is the greatest one.*

Proof. We prove the limit inferior claim, the other part can be proved analogously. First, we claim that $a = \liminf x_n$ is an accumulation point. Indeed, using the notation above, $a = \lim a_n$, hence given any $\epsilon > 0$, for $n > n_0$, we have $a - \epsilon < a_n < a + \epsilon$. In particular, choose $n_1 > n_0$, then $a - \epsilon < a_{n_1} < a + \epsilon$. Therefore, for $n > n_1$ we have $a_{n_1} \leq x_n < a + \epsilon$. We conclude that $a - \epsilon < x_n < a + \epsilon$, by theorem 130, a is an accumulation point. To prove the minimality, let $c < a$. We claim c is not an accumulation point. Since $c < a \Rightarrow c < a_{n_0}$, for some $n_0 \in \mathbb{N}$. Hence, $c < a_{n_0} \leq x_n$ for $n \geq n_0$. Finally, setting $\epsilon = a_{n_0} - c$, we conclude that the interval $(c - \epsilon, c + \epsilon)$ doesn't contain any x_n for $n > n_0$, by theorem 130 this concludes the proof. □

Corollary 136. *(Bolzano–Weierstrass theorem) Every bounded sequence* x_n *has a convergent subsequence.*

Proof. Since x_n is bounded, $a = \liminf x_n$ is well defined and is an accumulation point. In particular, there's a subsequence of x_n converging to a. □

Corollary 137. *A sequence* x_n *is convergent if and only if* $\liminf x_n = \limsup x_n$ *(*x_n *has a unique accumulation point)*

Proof. If x_n is convergent, all subsequences converge to the same limit, in particular $\liminf x_n = \limsup x_n = \lim x_n$. Conversely, suppose $a = \liminf x_n = \limsup x_n$. Then, using the notation above, we can find n_0 such that $a - \epsilon < a_{n_0} \leq a \leq b_{n_0} < a + \epsilon$ and $n > n_0$ implies $a_{n_0} \leq x_n \leq b_{n_0}$. We conclude that $a - \epsilon < x_n < a + \epsilon$. □

Corollary 138. *If* $c < \liminf x_n$ *then* $\exists n_0 \in \mathbb{N}$ *such that* $n > n_0 \Rightarrow c < x_n$*. Similarly, if* $c > \limsup x_n$ *then* $\exists n_1 \in \mathbb{N}$ *such that* $n > n_1 \Rightarrow c > x_n$*.*

17 Cauchy Sequences

A sequence x_n is called a **Cauchy sequence** if given $\epsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that for $n, m > n_0$ we have

$$|x_n - x_m| < \epsilon$$

In other words, a Cauchy sequence is a sequence such that its terms x_n are infinitely close for sufficiently large n. It's reasonable to expect that a sequence with this property converges, and that is indeed true as the theorem below shows *(for sequences in $\mathbb{R}$, we will see in a few weeks when we talk about topology, that it's possible to construct a topological space where no Cauchy sequence converges.)*

Theorem 139. *Every Cauchy sequence is convergent.*

The proof is a direct consequence of the two lemmas below.

Lemma 140. *Every Cauchy sequence is bounded.*

Proof. By definition, we can find $n_0 \in \mathbb{N}$ such that $m, n > n_0 \Rightarrow |x_n - x_m| < 1$. Fix x_m and set $M := \max\{|x_1|, |x_2|, \dots, |x_{n_0}|, |x_m - 1|, |x_m + 1|\}$, then $x_n \in [-M, M]$. □

Lemma 141. *If a Cauchy sequence x_n has a convergent subsequence x_{n_k} with $\lim_{k\to\infty} x_{n_k} = a$ then it converges and $\lim x_n = a$.*

Proof. Given $\epsilon > 0$, it's possible to find n_0 such that $m, n > n_0 \Rightarrow |x_n - x_m| < \frac{\epsilon}{2}$. Additionally, it's possible to find m_0 such that $n_k > m_0 \Rightarrow |x_{n_k} - a| < \frac{\epsilon}{2}$, take one $n_k > n_0$ such that this is true. Then $n > n_0 \Rightarrow |x_n - a| < |x_n - x_{n_k}| + |x_{n_k} - a| < \epsilon$. □

Now we prove the converse of the theorem above.

Theorem 142. *Every convergent sequence is a Cauchy sequence.*

Proof. Suppose $a := \lim x_n$. Then it's possible to find n_0 and n_1 such that $n > n_0 \Rightarrow |x_n - a| < \frac{\epsilon}{2}$ and $m > n_1 \Rightarrow |x_m - a| < \frac{\epsilon}{2}$. We conclude that

$$|x_n - x_m| < |x_n - a| + |x_m - a| < \epsilon,$$

for $m, n > \max\{n_0, n_1\}$. □

We conclude that

Corollary 143. *A sequence x_n of real numbers is a Cauchy sequence if and only if it converges.*

18 Infinite limits

A divergent sequence x_n *converges to infinity*, denoted by $\lim x_n = +\infty$, if for any number $M > 0$, there is $n_0 > 0$ such that $n > n_0 \Rightarrow x_n > M$. Similarly, A sequence x_n *converges to negative infinity*, denoted by $\lim x_n = -\infty$, if for any number $M > 0$, there is $n_0 > 0$ such that $n > n_0 \Rightarrow x_n < -M$.

Example 144. *The sequence $x_n = n$ converges to infinity, since given any $M > 0$, take any natural number $n_0 > M$, then $x_n = n > M$ if $n > n_0$. On the other hand, the sequence $x_n = (-1)^n n$ is divergent but doesn't converge to ∞, nor to $-\infty$, since it is unbounded from above and below, and as a consequence of the definition a sequence converges, say to $+\infty$, then it's bounded from below, and similarly, converges to $-\infty$, then it's bounded from above.*

The following theorem, similar to theorem 123 gives some properties of infinite limits. The proof will be omitted.

Theorem 145. *(Arithmetic operations with infinite limits)*

1. *If $\lim x_n = +\infty$ and y_n is bounded from below, then $\lim(x_n + y_n) = +\infty$ and $\lim(x_n \cdot y_n) = +\infty$;*
2. *If $x_n > 0$ then $\lim x_n = 0$ if and only if $\lim \frac{1}{x_n} = +\infty$;*
3. *Let $x_n, y_n > 0$ be positive sequences. Then:*
   (a) *If x_n is bounded from below and $\lim y_n = 0$ then $\lim \frac{x_n}{y_n} = +\infty$;*
   (b) *If x_n is bounded and $\lim y_n = +\infty$ then $\lim \frac{x_n}{y_n} = 0$.*

Example 146. *Let $x_n = \sqrt{n+1}$ and $y_n = -\sqrt{n}$. Then $\lim x_n = \infty$,$\lim y_n = -\infty$. We have:*

$$\lim(x_n + y_n) = \lim \sqrt{n+1} - \sqrt{n} = \lim \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \lim \frac{1}{\sqrt{n+1} + \sqrt{n}},$$

*which gives $\lim(x_n + y_n) = 0$. However, it's **not true in general** that $\lim(x_n + y_n) = \lim x_n + \lim y_n$ if both sequences have infinite limit. For example, $x_n = n^2$ and $y_n = -n$ give a counter-example, since $\lim x_n = +\infty$, $\lim y_n = -\infty$, but $\lim(x_n + y_n) = +\infty$.*

Example 147. *Let* $x_n = [2 + (-1)^n]n$ *and* $y_n = n$*. Then* $\lim x_n = \lim y_n = +\infty$*, but* $\lim \frac{x_n}{y_n} = \lim[2 + (-1)^n]$ *doesn't exists. So it's not true in general that* $\lim \frac{x_n}{y_n} = 1$ *if* $\lim x_n = \lim y_n = +\infty$*.*

Example 148. *Let* $a > 1$*. Then* $\lim \frac{a^n}{n} = +\infty$*. Indeed,* $a = 1+s$ *with* $s > 0$*, so* $a^n = (1 + s)^n \geq 1 + ns + \frac{n(n-1)}{2}s^2$ *for* $n \geq 2$*, but* $\lim \frac{1+ns+\frac{n(n-1)}{2}s^2}{n} = +\infty$*, hence* $\lim \frac{a^n}{n} = +\infty$*. Arguing by induction, it's easy to show that for any* $m \in \mathbb{N}$*,* $\lim \frac{a^n}{n^m} = +\infty$*.*

Example 149. *Let* $a > 0$*. Then* $\lim \frac{n!}{a^n} = +\infty$*. Indeed, pick* $n_0 \in \mathbb{N}$ *such that* $\frac{n_0}{a} > 2$*. Then*

$$\frac{n!}{a^n} = \frac{n(n-1)\dots(n_0+1)n_0!}{a^{n_0}\underbrace{a\dots a}_{n-n_0}} > \frac{n_0!}{a^{n_0}}2^{n-n_0},$$

and it follows that $\lim \frac{n!}{a^n} = +\infty$*.*

19 Series

Given a sequence of real numbers x_n, the purpose of this section if to give meaning to expressions of the form, $x_1 + x_2 + x_3 + \dots$, that is, the formal sum of all the elements of the sequence x_n.

A natural way of doing this is to set $s_n := x_1 + \dots + x_n$, called *partial sums*, and define

$$\sum_{n=1}^{\infty} x_n := \lim s_n$$

It's a common practice to write $\sum x_n$ instead of $\sum\limits_{n=1}^{\infty} x_n$, and to call x_n the general term of the series. In these notes we shall adopt these conventions.

Since we define $\sum x_n$ as a limit, it may or may not exist. In case $\sum x_n = L \in \mathbb{R}$ we say that the series $\sum x_n$ converges, otherwise we say $\sum x_n$ diverges.

Theorem 150. *If the series* $\sum x_n$ *converges then* $\lim x_n = 0$*.*

Proof. Indeed, we have $x_n = s_n - s_{n-1}$. Therefore, $\lim x_n = \lim(s_n - s_{n-1}) = \lim s_n - \lim s_{n-1} = 0$. □

The converse of the theorem above is not true. Here's a counterexample:

Example 151. *(harmonic series) Consider the series* $\sum \frac{1}{n}$. *We obviously have* $\lim \frac{1}{n} = 0$, *however, we claim* $\sum \frac{1}{n}$ *diverges. Indeed, in order to prove that* $\lim s_n$ *diverges, it's enough to find a divergent subsequence. Take for example* s_{2^n}:

$$\begin{aligned} s_{2^n} &= 1 + \frac{1}{2} + \ldots + \frac{1}{2^n} \\ &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots \\ &> 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \frac{8}{16} + \ldots + \frac{2^{n-1}}{2^n} \\ &= 1 + n \cdot \frac{1}{2} \end{aligned}$$

Hence, $s_{2^n} > 1 + n \cdot \frac{1}{2}$ *and* $\lim s_{2^n} = +\infty$.

Example 152. *(geometric series) The series* $\sum a^n$, *with* $a \in \mathbb{R}$, *diverges if* $|a| \geq 1$, *since the general term* $x_n = a^n$ *doesn't satisfy* $\lim x_n = 0$. *If* $|a| < 1$, *then* $\sum a^n$ *converges. Indeed, we can show by induction that*

$$s_n = \frac{1 - a^{n+1}}{1 - a},$$

and hence $\sum a^n = \lim s_n = \frac{1}{1-a}$, *if* $|a| < 1$.

Theorem 153. *Given series* $\sum a_n, \sum b_n$, *we have:*

1. *If* $\sum a_n$ *and* $\sum b_n$ *converge, then* $\sum(a_n + b_n)$ *converges and* $\sum(a_n + b_n) = \sum a_n + \sum b_n$.

2. *Let* $c \in \mathbb{R}$. *If* $\sum a_n$ *converges, then* $\sum c\, a_n$ *also converges, and* $\sum c\, a_n = c \sum a_n$.

3. *Suppose* $\sum a_n$ *and* $\sum b_n$ *converge, set* $c_n := \sum\limits_{i=1}^{n} a_i b_n + \sum\limits_{j=1}^{n-1} a_n b_j$. *Then* $\sum c_n$ *converges and* $\sum c_n = (\sum a_n) \cdot (\sum b_n)$.

Example 154. *(telescoping series) The series* $\sum \frac{1}{n(n+1)}$ *is convergent. Since* $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, *we easily see that* $s_n = 1 - \frac{1}{n+1}$, *so* $\sum \frac{1}{n(n+1)} = 1$.

Example 155. *The series* $\sum(-1)^n$ *is divergent since the sequence* $(-1)^n$ *has two distinct accumulation points, so it's impossible to have* $\lim(-1)^n = 0$.

Theorem 156. *Let* $a_n \geq 0$ *be a nonnegative sequence of real numbers. Then* $\sum a_n$ *converges if and only if the partial sum* s_n *is a bounded sequence for every* $n \in \mathbb{N}$.

Proof. The implication is clear. The converse follows from the fact that every bounded monotone sequence converges. □

Corollary 157. *(Comparison principle) Suppose* $\sum a_n$ *and* $\sum b_n$ *are series of nonnegative real numbers, i.e.* $a_n, b_n \geq 0$. *If there are* $c \in \mathbb{R}$ *and* $n_0 \in \mathbb{N}$ *such that* $a_n \leq c\, b_n$ *for* $n > n_0$, *then if* $\sum b_n$ *converges,* $\sum a_n$ *converges. Moreover, if* $\sum a_n$ *diverges then* $\sum b_n$ *diverges.*

Example 158. *If* $r > 1$, *the series* $\sum \frac{1}{n^r}$ *converges. Indeed, the general term of this series is positive, so the partial sums* s_n *are increasing, hence it's enough to prove that a subsequence of* s_n *is bounded. We claim* s_{2^n-1} *is bounded. We have:*

$$\begin{aligned}
s_{2^n-1} &= 1 + \frac{1}{2^r} + \ldots + \frac{1}{(2^n-1)^r} \\
&= 1 + \left(\frac{1}{2^r} + \frac{1}{3^r}\right) + \left(\frac{1}{4^r} + \frac{1}{5^r} + \frac{1}{6^r} + \frac{1}{7^r}\right) + \ldots + \frac{1}{(2^n-1)^r} \\
&< 1 + \frac{2}{2^r} + \frac{4}{4^r} + \frac{8}{8^r} + \ldots + \frac{2^{n-1}}{2^{(n-1)r}} \\
&= \sum_{j=0}^{n-1} \left(\frac{2}{2^r}\right)^j
\end{aligned}$$

On the other hand, the geometric series $\sum\limits_{j=0}^{\infty} \left(\frac{2}{2^r}\right)^j$ *converges since* $\frac{2}{2^r} < 1$. *We conclude that* s_{2^n-1} *is bounded and the claim follows.*

Corollary 159. *(Cauchy's criteria) The series* $\sum a_n$ *is convergent if and only if given* $\epsilon > 0$, *there is* $n_0 \in \mathbb{N}$ *such that* $|a_{n+1} + \ldots + a_{n+p}| < \epsilon$ *for* $n > n_0$.

Proof. Notice that s_n converges if and only if it is a Cauchy sequence (see Corollary 143). □

A series $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ is convergent. A series with all of its terms positive (or negative) is convergent if and only if is absolutely convergent. Hence, in this case the two notion coincide. Here's a classical counterexample that shows that they don't coincide in general:

Example 160. *Consider the series* $\sum \frac{(-1)^n}{n}$. *We already know that* $\sum \frac{1}{n}$ *diverges, however we claim that* $\sum \frac{(-1)^n}{n}$ *converges. Indeed, notice that the subsequence* s_{2n} *satisfies*

$$s_2 < s_4 < s_6 < \ldots < s_{2n},$$

and is a Cauchy sequence, hence convergent. Whereas s_{2n-1} *satisfies*

$$s_1 > s_3 > s_5 > \ldots > s_{2n-1},$$

so it's bounded and monotone, hence convergent as well. Set $a := \lim s_{2n}$, $b := \lim s_{2n-1}$, *then since* $s_{2n} - s_{2n-1} = \frac{1}{2n} \to 0$, *we necessarily have* $a = b$. *We conclude that* s_n *has only one accumulation point, hence converges. (We will see later that* $a = b = \log 2$)

A series $\sum a_n$ is **conditionally convergent** if $\sum a_n$ is convergent but $\sum |a_n|$ is divergent. The example above shows that $\sum \frac{(-1)^n}{n}$ is conditionally convergent.

Theorem 161. *Every absolutely convergent series* $\sum a_n$ *is convergent.*

Proof. By hypothesis, $\sum a_n$ is Cauchy, so we can find $n_0 \in \mathbb{N}$ such that $n > n_0, \forall p \in \mathbb{N} \Rightarrow |a_{n+1}| + \ldots + |a_{n+p}| < \epsilon$. In particular, $|a_{n+1} + \ldots + a_{n+p}| < |a_{n+1}| + \ldots + |a_{n+p}| < \epsilon$, the conclusion follows from Cauchy's criteria (Corollary 159). □

Corollary 162. *Let* $\sum b_n$ *a convergent series with* $b_n \geq 0$. *If there are* $n_0 \in \mathbb{N}$ *and* $c \in \mathbb{R}$ *such that* $n > n_0 \Rightarrow |a_n| \leq c\, b_n$ *then the series* $\sum a_n$ *is absolutely convergent.*

Corollary 163. *(The root test) If there are* $n_0 \in \mathbb{N}$ *and* $c \in \mathbb{R}$ *such that* $n > n_0 \Rightarrow \sqrt[n]{|a_n|} \leq c < 1$, *then the series* $\sum a_n$ *is absolutely convergent. In other words, if* $\limsup \sqrt[n]{|a_n|} < 1$ *then* $\sum a_n$ *is absolutely convergent. On the other hand, if* $\limsup \sqrt[n]{|a_n|} > 1$, *then* $\sum a_n$ *diverges.*

Proof. In this case, we can compare $\sum |a_n|$ with $\sum c^n$, the latter (absolutely) converges since it's a geometric series with $0 < c < 1$. If $\sqrt[n]{|a_n|} > 1$ for n sufficiently large, then $\lim a_n \neq 0$. $\square$

Corollary 164. *(The root test – second version) If $\lim \sqrt[n]{|a_n|} < 1$, then the series $\sum a_n$ is absolutely convergent. If $\lim \sqrt[n]{|a_n|} > 1$, then the series $\sum a_n$ is divergent.*

Example 165. *Let $a \in \mathbb{R}$ and consider the series $\sum na^n$. Notice that $\lim \sqrt[n]{n\,|a|^n} = \lim \sqrt[n]{n} \lim |a| = |a|$. Hence, if $|a| < 1$ the series $\sum na^n$ is absolutely convergent and if $|a| > 1$ it diverges. If $|a| = 1$ the series also diverges, since $\lim na^n \neq 0$ in this case.*

Theorem 166. *(The ratio test) Let $\sum a_n$ and $\sum b_n$ be series of real numbers such that $a_n \neq 0, b_n > 0, \forall n \in \mathbb{N}$ and $\sum b_n$ convergent. If there is $n_0 \in \mathbb{N}$ such that $n > n_0 \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| \leq \left|\frac{b_{n+1}}{b_n}\right|$, then $\sum a_n$ is absolutely convergent.*

Proof. Consider the inequalities:

$$\left|\frac{a_{n_0+2}}{a_{n_0+1}}\right| \leq \left|\frac{b_{n_0+2}}{b_{n_0+1}}\right|$$
$$\left|\frac{a_{n_0+3}}{a_{n_0+2}}\right| \leq \left|\frac{b_{n_0+3}}{b_{n_0+2}}\right|$$
$$\cdots$$
$$\left|\frac{a_n}{a_{n-1}}\right| \leq \left|\frac{b_n}{b_{n-1}}\right|$$

Multiplying them together, we have:

$$\left|\frac{a_n}{a_{n_0+1}}\right| \leq \left|\frac{b_n}{b_{n_0+1}}\right|$$

Hence, $|a_n| \leq c\, b_n$ and the result follows by the comparison principle. $\square$

Corollary 167. *(The ratio test – second version) If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series $\sum a_n$ is absolutely convergent. If $\limsup \left|\frac{a_{n+1}}{a_n}\right| > 1$, then the series $\sum a_n$ is divergent.*

Proof. For the convergence, take $b_n = (\limsup \left|\frac{a_{n+1}}{a_n}\right|)^n$ in theorem 166. If $\limsup \left|\frac{a_{n+1}}{a_n}\right| > 1$ then $\lim a_n \neq 0$. □

Corollary 168. *(The ratio test – third version) If* $\lim \left|\frac{a_{n+1}}{a_n}\right| < 1$ *then* $\sum a_n$ *is absolutely convergent, if* $\lim \left|\frac{a_{n+1}}{a_n}\right| > 1$ *then* $\sum a_n$ *diverges.*

Example 169. *Fix* $x \in \mathbb{R}$ *and consider the series* $\sum \frac{x^n}{n!}$*, then* $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{n+1} \to 0$ *regardless of* x*, and the series is absolutely convergent. We will see later that this series coincides with* e^x*.*

Theorem 170. *(Root test is stronger than the ratio test) For any bounded sequence* a_n *of positive numbers we have*

$$\liminf \frac{a_{n+1}}{a_n} \leq \liminf \sqrt[n]{a_n} \leq \limsup \sqrt[n]{a_n} \leq \limsup \frac{a_{n+1}}{a_n},$$

In particular, if $\lim \frac{a_{n+1}}{a_n} = c$ *then* $\lim \sqrt[n]{a_n} = c$*.*

Proof. It's enough to prove that $\limsup \sqrt[n]{a_n} \leq \limsup \frac{a_{n+1}}{a_n}$, the first inequality can be proven *mutatis mutandis*. We argue by contradiction, suppose there is a $k \in \mathbb{R}$ such that

$$\limsup \sqrt[n]{a_n} > k > \limsup \frac{a_{n+1}}{a_n}$$

Proceeding as in the proof of theorem 166, we can find $n_0 \in \mathbb{N}$ such that $n > n_0 \Rightarrow a_n < c\,k^n$, which implies that $\sqrt[n]{a_n} < c^{\frac{1}{n}}\,k$ and hence:

$$\limsup \sqrt[n]{a_n} \leq k$$

a contradiction. □

Example 171. *A nice application of the theorem above is the computation of* $\lim \frac{n}{\sqrt[n]{n!}}$*. Set* $x_n = \frac{n}{\sqrt[n]{n!}}$ *and* $y_n = \frac{n^n}{n!}$*, then* $x_n = \sqrt[n]{y_n}$*. On the other hand,* $\frac{y_{n+1}}{y_n} = (1 + \frac{1}{n})^n$*, hence* $\lim \frac{y_{n+1}}{y_n} = e$*, and it follows that* $\lim \frac{n}{\sqrt[n]{n!}} = e$*.*

Example 172. *Given two distinct numbers* $a, b \in \mathbb{R}$*, consider the sequence* $x_n = \{a, ab, a^2b, a^2b^2, a^3b^2, \ldots\}$*, then the ratio* $\frac{x_{n+1}}{x_n} = b$ *if* n *is odd, and* $\frac{x_{n+1}}{x_n} = a$ *if* n *is even, hence the sequence* $\frac{x_{n+1}}{x_n}$ *doesn't converge and* $\lim \frac{x_{n+1}}{x_n}$ *doesn't exist. On the other hand, we have* $\lim \sqrt[n]{x_n} = \sqrt{ab}$*. This demonstrates that in the theorem above the inequalities can be strict.*

Theorem 173. *(Dirichlet) Let b_n be a nonincreasing sequence of positive numbers with $\lim b_n = 0$, and $\sum a_n$ be a series such that the partial sum s_n is a bounded sequence. Then the series $\sum a_n b_n$ converges.*

Proof. Notice that

$$\begin{aligned} a_1 b_1 + a_2 b_2 + \ldots + a_n b_n &= a_1(b_1 - b_2) + (a_1 + a_2)(b_2 - b_3) + \\ &+ (a_1 + a_2 + a_3)(b_3 - b_4) + \ldots + (a_1 + \ldots + a_n) b_n \\ &= \sum_{i=2}^{n} s_{i-1}(b_{i-1} - b_i) + s_n b_n \end{aligned}$$

Since s_n is bounded, say $|s_n| \leq k$ and $b_n \to 0$, we have $\lim s_n b_n = 0$. Moreover, $|\sum_{i=2}^{n} s_{i-1}(b_{i-1} - b_i)| \leq k|\sum_{i=2}^{n}(b_{i-1} - b_i)| = k(b_1 - b_n)$. So $\sum_{i=2}^{n} s_{i-1}(b_{i-1} - b_i)$ converges, and therefore, by comparison, $\sum a_n b_n$ converges as well. □

We can weaken the hypothesis $\lim b_n = 0$. Indeed, if $\lim b_n = c$ just take $b_n^* := b_n - c$ and use this new sequence instead. We conclude:

Corollary 174. *(Abel) If $\sum a_n$ is convergent and b_n is a nonincreasing sequence of positive numbers then $\sum a_n b_n$ converges.*

Corollary 175. *(Leibniz) Let b_n be a nonincreasing sequence of positive numbers with $\lim b_n = 0$. Then the series $\sum (-1)^n b_n$ converges.*

Proof. In this case, $a_n = (-1)^n$ has bounded partial sum, namely $|s_n| \leq 1$, and the result follows directly from theorem 173. □

Example 176. *Some periodic real valued functions can be written as a linear combination of $\sum \cos(nx)$ and $\sum \sin(nx)$. The properties of such functions and generalizations are addressed in area of mathematics called **Fourier Analysis**. E. Stein's book on the subject is a wonderful first-read of the topic.*

Take the example of $f(x) = \sum \frac{\cos(nx)}{n}$, we claim that if $x \neq 2\pi k$, $k \in \mathbb{Z}$ then $f(x)$ is well-defined, i.e. $\sum \frac{\cos(nx)}{n}$ converges. Indeed, let $a_n = \cos(nx)$ and $b_n = \frac{1}{n}$, then b_n is decreasing, so by theorem 173, it's enough to prove that the partial sums s_n of $\sum a_n$ are bounded. In other words, we need to show that

$$s_n = \cos(x) + \cos(2x) + \cos(3x) + \ldots + \cos(nx)$$

is bounded. Recall, that $e^{ix} = \cos(x) + i\sin(x)$. *Therefore:*

$$1 + s_n = Re[1 + e^{ix} + e^{2ix} + e^{3ix} + \ldots + e^{nix}]$$
$$1 + s_n = Re[\frac{1 - e^{(n+1)ix}}{1 - e^{ix}}]$$
$$1 + s_n \leq \frac{2}{|1 - e^{ix}|}$$

It follows that s_n *is bounded and we conclude that* $\sum \frac{\cos(nx)}{n}$ *converges if* $x \neq 2\pi k$.

Let $f : \mathbb{N} \to \mathbb{N}$ be a bijection and $\sum a_n$ be a series of real numbers. Set $b_n = a_{f(n)}$. We say $\sum a_n$ is **commutatively convergent** if $\sum a_n = \sum b_n$ for every bijection $f : \mathbb{N} \to \mathbb{N}$. We will show below that the notion of commutative convergence coincides with absolute convergence.

Theorem 177. *A series* $\sum a_n$ *is absolutely convergent if and only if is commutatively convergent.*