Monday, September 9, 2024 3:34 PM

$I_m = \{n \in \mathbb{N};\ 1 \le n \le m\}$

$f: I_n \to X$; $|X| = n$ (→ number of elements), $n(X)$, ~~#X~~ $\quad I_1 = \{1\}$, $\mathcal{P}(I_1) = \{\emptyset, \{1\}\}$

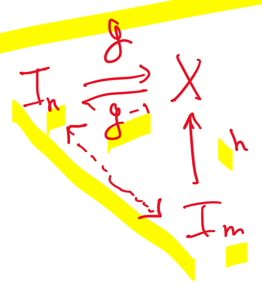
Theorem: Let $X \subseteq I_n$, then if there is a bijection $f: I_n \to X$ then $X = I_n$.

Proof: We prove this result by induction on "n". The result is trivial if $n=1$, so assume it is valid for I_n, we show that it also valid for I_{n+1}. Take $X \subseteq I_{n+1}$ and there is a bijection $f: I_{n+1} \to X$. Take $a = f(n+1)$ and consider $f|_{I_n}: I_n \to X - \{a\}$, if $X - \{a\} \subseteq I_n$ then $X - \{a\} = I_n$ so $a = n+1$ and $X = I_{n+1}$. Suppose $X - \{a\} \not\subseteq I_n$, then $n+1 \in X - \{a\}$. Let $f(b) = n+1$, then we define a bijection $g: I_{n+1} \to X$ given by $g(x) = f(x)$ if $x \notin \{n+1, b\}$ and $g(n+1) = n+1$ $g(b) = a$. Now, $g: I_n \to X - \{n+1\}$ but $X - \{n+1\} \subseteq I_n$, by induction $X = I_{n+1}$. □

Corollary: If there is a bijection $f: I_n \to I_m$ then $m = n$. In particular, if $g: I_n \to X$, $h: I_m \to X$ are bijections then $m = n$.

$n > m \qquad n < m$

$I_m \subseteq I_n \qquad I_n \subseteq I_m$



Corollary: There is no bijection $f: X \to Y$ between a finite set and a proper subset $Y \subsetneq X$.

Proof: Suppose that there is a bijection $f: X \to Y$

$g^{-1}(Y) = A$

$$\begin{array}{ccc} f: X & \xrightarrow{\simeq} & Y \\ g \uparrow \simeq & & \simeq \uparrow g|_A \\ A \subseteq I_n & \xrightarrow{\simeq} & A \end{array}$$

Then the composition $g|_A^{-1} \circ f \circ g: I_n \to A$ defines a bijection between I_n and a proper subset $A \subsetneq I_n$, a contradiction. □

Theorem: Let X be finite and $Y \subseteq X$, then Y is also finite and $|Y| \le |X|$, the equality occurs only if $X = Y$.

$X = I_n$

$f: X \to f(X) \subseteq Y$, I_n

Corollary: Suppose Y is finite and $f: X \to Y$ is an injective function. Then X is also finite and $|X| \le |Y|$.

f injective $\iff$ left-inverse g $(g \circ f)(x) = x$

g surjective $\iff$ right-inverse f $(g \circ f)(x) = x$

Corollary: $f: X \to Y$ surjective, X finite

$g: Y \to X$

⇓

Y is also finite

A set X is infinite if it's not finite.

Ex: (Natural numbers) $\mathbb{N}$

$f: I_n \to \mathbb{N}$

Consider $M = f(1) + f(2) + \dots + f(n)$ then $M \notin f(I_n)$. So there is no bijection from I_n to $\mathbb{N}$ regardless how big "n" is.

Ex: $\mathbb{Z}$, $\mathbb{Q}$

Let $X \subseteq Y$ if X is infinite $\Rightarrow$ Y is infinite

$f: I_n \to Y$

$f: f^{-1}(X) \to X$

A set $X \subseteq \mathbb{N}$ is <u>bounded</u> if $x \le M$ ($\in \mathbb{N}$) for every $x \in X$.

Theorem: Let $X \subseteq \mathbb{N}$ then the following are equivalent:

a. X is finite;

b. X is bounded;

c. X has a greatest element.

100

$X \subseteq I_{100}$

p_r $X \subseteq I_p$

Proof: $a \to b$, $b \to c$, $c \to a$

$a \to b$ $X = \{x_1, x_2, \dots, x_n\}$, $M = x_1 + x_2 + \dots + x_n$ is a bound

$b \to c$ $A = \{n;\ n \ge x\ \forall x \in X\}$

$c \to a$ Immediate

P. Halmos — Naive set theory

Cardinality

Proposition X, Y finite $X \cap Y = \emptyset$ then

$|X \cup Y| = |X| + |Y|$

$f: I_n \to X$

$g: I_m \to Y$

$I_{n+m} \xrightarrow{\simeq} X \cup Y$

Corollary: $X_i \cap X_j = \emptyset$

$|\bigcup X_i| = \sum |X_i|$

Countable < finite / $f: \mathbb{N} \to X$

Countable sets

A set X is countable if there is a bijection

$f: \mathbb{N} \to X$ (countably infinite)

or X is finite.

$\{x_1, x_2, x_3, x_4, x_5, x_6, \dots\}$

Ex: $X = \{n;\ n \text{ even}\}$

$f: \mathbb{N} \to \mathbb{N}$ $\quad f(\mathbb{N}) = X$

$x \mapsto 2x$

$Y = \{n;\ n \text{ is odd}\}$

$f(x) = 2x + 1$

Theorem: Let X be an infinite set. Then X has a countably infinite subset.

Proof: It is enough to find an injective function:

$f: \mathbb{N} \to X$

Choose an element $a_1 \in X$, $X_1 = X - \{a_1\}$

$f(1) = a_1$

$a_2 \in X_1$

$f(2) = a_2$

⋮

$a_n \in X_{n-1}$

$f(n) = a_n$

⋮