Wednesday, September 4, 2024 3:38 PM

Finite sets, coutable and uncountable sets

The natural numbers $\mathbb{N}$

$f(x) = f(y) \Rightarrow x = y$

We start with a set $\mathbb{N}$, whose elements will be called "natural numbers", and we assume that there is a function $s: \mathbb{N} \to \mathbb{N}$ with the following properties

Axiom 1 $s(n)$ is injective.

Axiom 2: $\mathbb{N} - s(\mathbb{N})$ consists of one element, say "1".

Axiom 3: (Principle of Induction) If $X \subseteq \mathbb{N}$ with the property that $1 \in X$ and $n \in X \Rightarrow s(n) \in X$, then $X = \mathbb{N}$.

$1 \in X$?

Ex: Claim: $s(n) \neq n$.

Let $X = \{n \in \mathbb{N} \mid s(n) \neq n\}$, we want to show the $X = \mathbb{N}$. We will use Axiom 3.

By axiom 2, $1 \in X$. Suppose that $n \in X$, we claim that $s(n) \in X$.

↳ $s(n) \neq n$

$s(s(n)) \neq s(n)$

□

Ex: (Functions) Let $f: X \to X$

$f^1 = f$

$f^{s(n)} = f \circ f^n$ $\quad s(1) = 2$

$f^2 = f^{s(1)} = f \circ f$

$f^3 = \underbrace{f \circ f \circ f}_{3}$

$n + m = s^m(n)$ $\quad s^n(m) = s^m(n)$

$((m+1)+1)+\ldots = (n+1)+1.$

The sum $m + n \in \mathbb{N}$

$m + n := s^n(m)$ $\quad (((m+1)+1)+1)\ldots$

$n = 1$

$m + 1 = s(m)$

$m + (n+1) = (m+n) + 1$

$m + s(n) = s(m+n)$

Prop:
- $m + (n+p) = (m+n) + p$
- $m + n = n + m$
- $m + n = m + p \Rightarrow n = p$
- Given $m, n \in \mathbb{N}$ only one of the following happens:
  1) $m = n$
  2) $m = n + p$, $p \in \mathbb{N}$
  3) $n = m + p$, $p \in \mathbb{N}$

Order

"m is less than n"

$m < n$ or $n > m$

$n = m + p$, $p \in \mathbb{N}$

$n > m$

$1 < 2$ $\quad 7 < 10$

$2 = 1 + 1$ $\quad 10 = 7 + 3$

Properties: 1) $m < n$, $n < p$ then $m < p$

$n = m + K$, $p = n + q$ $\Rightarrow$ $p = (m+K) + q$

$p = m + (K + q)$

$m < p$. □

2) Given $m, n \in \mathbb{N}$, only one of the following happens:
- i) $m = n$
- ii) $m > n$
- iii) $m < n$

3.) $m < n \Rightarrow m + p < n + p$;

Multiplication

$f_m(n) = n + m$ $\quad f_m: \mathbb{N} \to \mathbb{N}$, $n \mapsto n + m$

$m \cdot 1 = m$

$m \cdot (n+1) = (f_m)^n(m)$

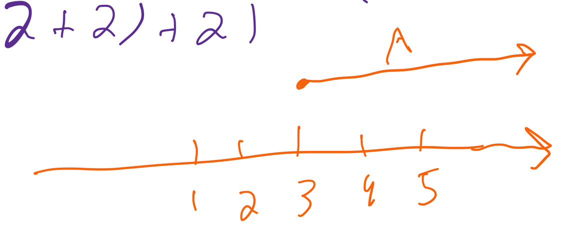
$m \cdot n + m = \underbrace{m + m + m + \ldots +}_{n} = (n+1)m$

$2 \cdot 1 = 2$

$2 \cdot 3 = (f_2)^2(2)$

$2 \cdot 3 = ((2+2)+2)$

Properties:
1) $m \cdot (n+p) = m \cdot n + m \cdot p$
2) $m \cdot n = n \cdot m$
3) $m < n \Rightarrow m \cdot p < n \cdot p$
4) $mp = np \Rightarrow m = n$



$X = \{2, 3, 10\}$

$\min X = 2$

$X = \{100, 150, 1000\}$

$\min X = 100$

$m \leq p \quad \forall p \in X$

Principle of Well-ordering.

$A \subseteq B$

Theorem: Given $A \subseteq \mathbb{N}$, $A \neq \emptyset$, then A has a minimum element

Proof: Consider the set $I_n = \{m \in \mathbb{N} \mid 1 \leq m \leq n\}$ and define $X = \{m \in \mathbb{N} \mid I_m \subset \mathbb{N} - A\}$. If $1 \in A$, then 1 is the minimum element. ($\mathbb{N} - A = A^c$)

Suppose $1 \notin A$, $1 \in X$, $X \neq \emptyset$, $X \neq \mathbb{N}$

$X \subseteq A^c$: $\mathbb{N} - A$

By the principle of induction, there is a $n \in \mathbb{N}$ such that $n \in X$ but $n+1 \notin X$, $a = n+1$ is the smallest element of "A".

□

Theorem (Strong induction): Let $X \subseteq \mathbb{N}$ having the following property: $\forall n \in \mathbb{N}$, X contains all $m < n \Rightarrow n \in X$. Then $X = \mathbb{N}$.

Proof: $Y = \mathbb{N} - X$, the claim is that $Y = \emptyset$. Suppose $Y \neq \emptyset$, then Y has a minimum element, say n_0, by hypothesis X contains $m < n_0 \Rightarrow n_0 \in X$, a contradiction.

Ex: (defining functions by induction/recurrence)

$f(1) = 1$

$f(n+1) = (n+1) \cdot f(n)$

$f(2) = 2 \cdot 1$

$f(3) = 3 \cdot f(2) = 3 \cdot 2 \cdot 1$

$f(4) = 4 \cdot f(3) = 4 \cdot 3 \cdot 2 \cdot 1$

$f(n) = n \cdot (n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$ "factorial"

Ex: $f(1) = 1$ $f(2) = 2$,

$f(n+2) = \dfrac{f(n) + f(n+1)}{2}$

$f(3) = \dfrac{f(1) + f(2)}{2} = \dfrac{1+2}{2} = \dfrac{3}{2}$

$f(4) = \dfrac{f(2) + f(3)}{2} = \dfrac{2 + \frac{3}{2}}{2} = \dfrac{7}{4}$

$\vdots$

Ex: Fibonacci sequence

$a_0 = 1$ $a_1 = 1$ $\quad 1, 1, 2, 3, 5, 7, 12, 19, 31, \ldots$

$a_{n+1} = a_n + a_{n-1}$

Finite Sets

$I_n = \{1, 2, 3, \ldots, n\}$

A set is finite if there is a bijection

$f: I_n \to X$

↳ number of elements of the set X

↳ counting function.

Ex: $\{a, b, c, d\} = X$
↑ ↑ ↑ ↑
1 2 3 4

Theorem: If $A \subseteq I_n$ and there is a bijection $f: I_n \to A$ then $A = I_n$.