Wednesday, August 28, 2024 3:16 PM

Cartesian Product

Given A, B we define: $(1,2) \neq (2,1)$

Zarmelo-Frenkel axioms

$A \times B = \{ (a,b) \mid a \in A, b \in B \}$

(a,b): ordered pair

coordinates: $(a,b) = (c,d) \Leftrightarrow a = c$ and $b = d$

Ex: $A = \{1,2\}$, $B = \{3,7\}$

$$A \times B = \begin{Bmatrix} (1,3), (1,7) \\ (2,3), (2,7) \end{Bmatrix}$$

Ex: $\mathbb{R}^2 = \{ (a,b) \mid a, b \in \mathbb{R} \} = \mathbb{R} \times \mathbb{R}$

$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}}_{n}$

Functions

Sets

A function $f: A \to B$ consists of:

1. The domain "A"
2. The codomain "B"
3. Rule that associates each element of A, a unique element of B.

Ex: Identity

$f: X \to X$

$f(x) = x$

Ex: Let X be the set of all triangles.

$A: X \to \mathbb{R}$

triangle $\mapsto$ Area

(triangle with sides 2, 1) $\mapsto 1$

Ex: Consider this relation:

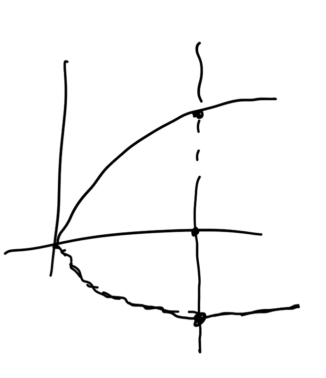
$f: \mathbb{Q} \to \mathbb{Q}$, $f(x) = \frac{1}{x}$ — not function

$f: \mathbb{Q} - \{0\} \to \mathbb{Q}$, $f(x) = \frac{1}{x}$ ✓

Not example:

$f: \mathbb{Z}_+ \to \mathbb{Z}$

$f(x) = \pm\sqrt{x}$



Perimeter: $\mathbb{Z}_+ \to$ Set of all triangles

$r \mapsto$ A triangle of perimeter "r"

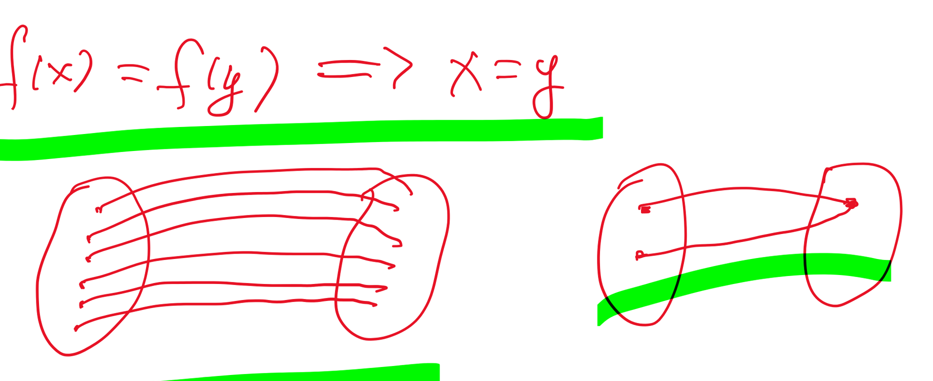
The graph of a function

$\Gamma(f) = \{ (x, f(x)) \mid x \in A \} \subseteq A \times B$

$f: A \to B$

A function $f: A \to B$ is injective (one-to-one):

$f(x) = f(y) \Rightarrow x = y$



A function $f: A \to B$ is surjective (onto):

$\forall b \in B \; \exists a \mid f(a) = b$

(∀: For every; ∃: there is; |: such that)

A function is bijective if it is both injective and surjective.

Ex: $f(x) = 3 \cdot x$; $f: \mathbb{Q} \to \mathbb{Q}$ $y = 3x \Rightarrow x = \frac{y}{3}$ $3x = \frac{1}{2}$, $x = \frac{1}{6}$

$f(x) = f(y) \Leftrightarrow \cancel{3}x = \cancel{3}y \Rightarrow x = y$

Injective $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$, $f(1) = f(-1) = 1$ but $1 \neq -1$

Not injective

Notice that there not x such that $f(x) < 0$, in particular $f(x) = x^2$ is not surjective.

Image of a function $f(x): A \to B$

$f(A) = \{ y \in B \mid y = f(x), x \in A \}$

Properties

1) $f(A \cup B) = f(A) \cup f(B)$
2) $X \subseteq Y \Rightarrow f(X) \subseteq f(Y)$ ✓
3) $f(\emptyset) = \emptyset$ ✓

Proof (2): $a \in f(X) \Rightarrow a = f(x)$ for some $x \in X$

$a = f(x)$ $x \in Y$

$a \in f(Y)$ □

The inverse image or pre-image

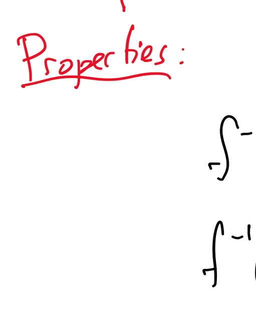
$f: A \to B$

$f^{-1}(C) = \{ a \in A \mid f(a) \in C \}$

Ex: $f(x) = x^2$ $f: \mathbb{R} \to \mathbb{R}$

$f^{-1}([1, +\infty)) = \{ a \in \mathbb{R} \mid a^2 \in [1, +\infty) \}$ $a^2 \geq 1$

$= \{ a \geq 1 \text{ or } a \leq -1 \}$



Properties:

$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

$f^{-1}(A^c) = [f^{-1}(A)]^c$

$f^{-1}(\emptyset) = \emptyset$

Composition of functions

$f: A \to B$, $g: B \to C$ $\Big\rangle (g \circ f): A \to C$

$(g \circ f)(x) = g(f(x))$

Ex: $f(x) = e^x$, $g(x) = x^2$

$(g \circ f)(x) = g(e^x) = (e^x)^2 = e^{2x}$

We say that g is a left (right) inverse of f if:

$g(f(x)) = x$

$(f(g(x)) = x)$

g is the inverse function of f if

$g(f(x)) = f(g(x)) = x$

Ex: $f(x) = e^x$, $f^{-1}(x) = \ln x$

$f(x) = \sqrt{x}$, $f^{-1}(x) = x^2$

$f(x) = \sin x$, $f^{-1}(x) = \sin^{-1} x = \arcsin x$

⋮

Proposition: A function f(x) has a left-inverse $\Leftrightarrow$ f(x) is injective.

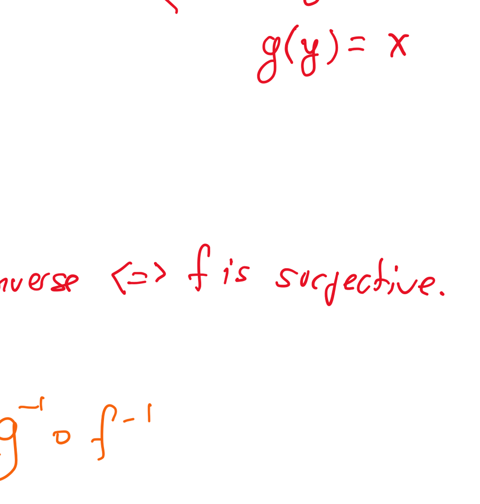
($\Rightarrow$) $f(x) = f(y)$ $g(f(x)) = x$

$g(f(x)) = g(f(y))$

$x = y$

($\Leftarrow$) $y = f(x)$ then set $g(y) = x$:

$g(f(x)) = x$



$y = f(x)$

$g(y) = x$

Proposition: $f: A \to B$ has a right inverse $\Leftrightarrow$ f is surjective.

Proof: Exercise.

$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

Families

We start with a set L (index set), and consider functions:

$f: L \to X$

We can represent $f(\ell)$ as $f_\ell \in X$.

Ex: $L = \{1, 2, 3\}$, $X = \mathbb{Q}$

$f: \{1, 2, 3\} \to \mathbb{Q}$

$(f(1), f(2), f(3))$

$L = \{1, 2, \dots, n\}$

$(f(1), \dots, f(n))$ a n-tuple

X is not necessarily a set of numbers, it could be a set whose elements are also sets:

$X_n = \{ x \in \mathbb{R} \mid x = n + 1 \}$

$X: \mathbb{N} \to P(\mathbb{R})$

$n \mapsto X_n$

$X_n = [n, n+1]$

$X: \mathbb{N} \to P(\mathbb{R})$

$1 \to [1, 2]$

$2 \to [2, 3]$

⋮