Monday, August 26, 2024 3:38 PM

Sets and functions

A set is a well-defined collection of elements.

By well-defined, I mean given an element "x", a collection "A" is a set if one can say whether or not "x" is in A.

Ex: $X = \{\text{triangles}\}$

$Y = \{\text{cars}\}$

$Z = \{\text{even numbers}\}$

$A = \{\text{students}\}$

$B = \{\text{phones}\}$

$C = \{\text{prime numbers}\}$

...

Notation

(x belongs to A)

$x \in A \Rightarrow$ x is an element of A

$x \notin A \Rightarrow$ x is not an element of A

$A = \{1, 2, 3\}$

$1 \in A, \; 5 \notin A, \; 3 \in A$

We denote a set by using capital letters A, B, C, ... and we describe a set by showing its elements.

$B = \{-1, 0, 10, \pi\}$

$\pi \in B, \; 2 \notin B$

Natural numbers

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

$f(x) = x \notin \mathbb{N}$

$1+i \notin \mathbb{N}$

$e \notin \mathbb{N}$

$\pi \notin \mathbb{N}$

$-1 \notin \mathbb{N}$

$0 \notin \mathbb{N}$

Integers

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$

$\mathbb{Z} \subseteq \mathbb{Q}$ (inclusion)

$i \notin \mathbb{Z}, \; \sqrt{2} \notin \mathbb{Z}, \; \frac{1}{2} \notin \mathbb{Z}, \ldots$

Rationals

$1+i \notin \mathbb{Q} \quad e \notin \mathbb{Q} \quad \pi \notin \mathbb{Q}$

$\mathbb{Q} = \{\text{fractions}\} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$

Set-builder notation

"The set of all x such that ..."

$A = \{x \mid x \text{ has property } P\}$

Ex: "x is a natural number bigger than 3"

$A = \{x \in \mathbb{N} \mid x > 3\}$

Definition: The empty set "$\emptyset$" is the set with no elements

$\emptyset = \{\}$

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Definition: Given two sets A, B we say A is a subset of B, i.e. $A \subseteq B$ if every element of A is in B.

Ex: $A = \{1, 2\}, \; B = \{1, 2, 3, 4\}$

$A \subsetneq B \qquad A \subseteq B \qquad A \subseteq A, \; B \subseteq B$

A proper subset $A \subsetneq B$ is a subset such that $A \neq B$.

Ex: $\{x \in \mathbb{Q} \mid x^2 + 1 = 0\} = \emptyset$

$\{x \in \mathbb{Q} \mid x^3 + 2 = 0\} = \emptyset$

$\{x \in \mathbb{Z} \mid \sqrt{x} = -2\} = \emptyset \qquad \emptyset \subseteq A$

Proposition: $\emptyset$ is a subset of any set.

$\emptyset \subseteq \mathbb{N}, \; \emptyset \subseteq \mathbb{Q}, \; \emptyset \subseteq \{x \in \mathbb{Q} \mid x^4 - e^x x + 1 = 0\}$

Ex: Power set of a set X

$\mathcal{P}(X) = \{A \mid A \subseteq X\}$

Ex: $\mathcal{P}(\{1, 2, 3\}) = \{\emptyset, \{1,2\}, \{1\}, \{1,3\}, \{2\}, \{2,3\}, \{3\}, \{1,2,3\}\}$ $\quad 2^3$

$|A| = \#A$

$A = \{1, 2, 3, \ldots, n\}$

$|\mathcal{P}(A)| = 2^n$

Properties

$A \subseteq A$

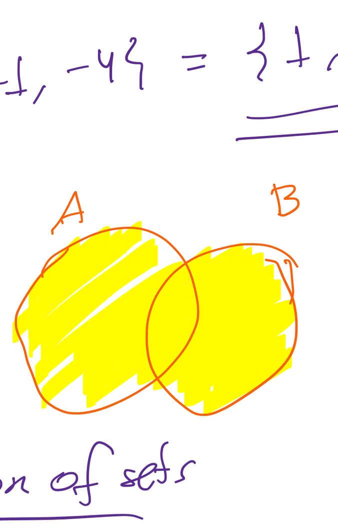
$A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$

$A \subseteq B$ and $B \subseteq A \Rightarrow A = B$

Union of sets

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

$\{1, 2\} \cup \{-1, -4\} = \{1, 2, -1, -4\}$

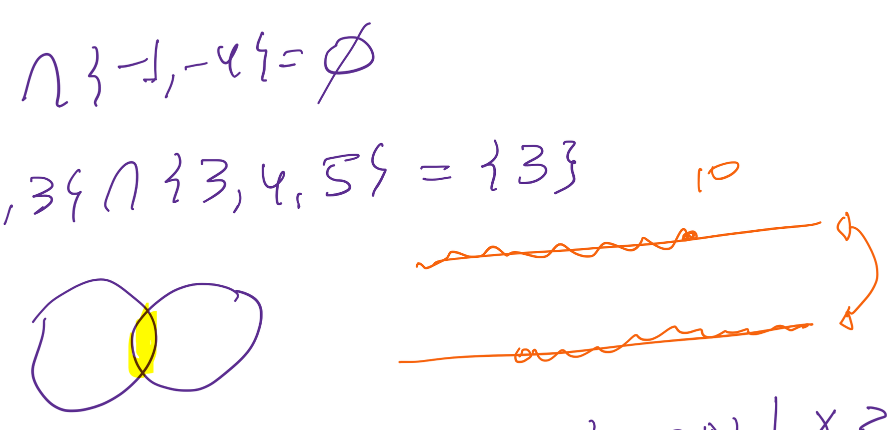


Intersection of sets

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Ex: $\{1, 2\} \cap \{-1, -4\} = \emptyset$

$\{1, 2, 3\} \cap \{3, 4, 5\} = \{3\}$



Ex: $A = \{x \in \mathbb{N} \mid x \leq 10\}, \; B = \{x \in \mathbb{N} \mid x > 5\}$

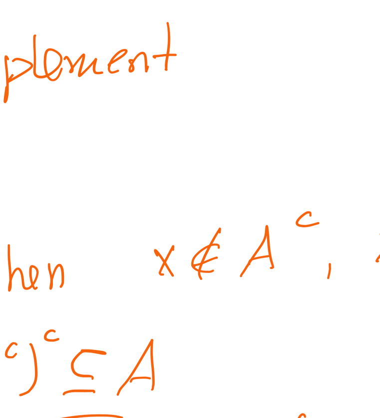
$A \cap B = \{x \in \mathbb{N} \mid 5 < x \leq 10\} = \{6, 7, 8, 9, 10\}$

$A \cup B = \mathbb{N} \qquad \{1, 2, 2\} = \{1, 2\} = \{1, 1, 2\}$

Properties

1) $A \cup \emptyset = A$
2) $A \cup A = A$
3) $A \cup B = B \cup A$
4) $(A \cup B) \cup C = A \cup (B \cup C)$
5) $A \cap \emptyset = \emptyset$
6) $A \cap A = A$
7) $A \cap B = B \cap A$
8) $(A \cap B) \cap C = A \cap (B \cap C)$
9) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A - B = \{x \mid x \in A \text{ and } x \notin B\}$
$= A \cap B^c$



$B^c = \{x \mid x \notin B\}$ (the complement)

$A \subseteq B$ and $B \subseteq A \Rightarrow A = B$

Properties of the complement

1. $(A^c)^c = A$

Proof: Take $x \in (A^c)^c$ then $x \notin A^c$, $x \in A$.

$(A^c)^c \subseteq A$

Conversely, take $x \in A$, then $x \notin A^c \Rightarrow x \in (A^c)^c$

$A \subseteq (A^c)^c$

$A = (A^c)^c$

2. $A \subseteq B \Rightarrow B^c \subseteq A^c$