Monday, October 7, 2024   3:38 PM

Sequences of real numbers

$x_n = x(n)$ $f(x)$ f_n

A sequence "x_n" is a function $x : \mathbb{N} \to \mathbb{R}$

$\{x_n\}_{n\in\mathbb{N}}$, x_n , $\{x_1, x_2, x_3, \dots\}$, (x_n)

x_n is bounded if:

$a \leq x_n \leq b$ $a, b \in \mathbb{R}$

$x(\mathbb{N}) \subseteq [a,b]$

otherwise, "x_n" is unbounded.

$x_n \leq b$ "bounded from above"

$a \leq x_n$ "bounded from below"

Given a sequence x_n and $K \subseteq \mathbb{N}$ infinite, we can build a new sequence with elements of x_n in the following way:

$\exists b : \mathbb{N} \to K$, since K is countable

$x_{n_k} : x \circ b$

$$\begin{array}{l} \qquad\qquad\qquad\qquad\quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \;\cdots \\ \text{Ex: } x_n = (-1)^n , \quad x_n = \{1, -1, 1, -1, 1, \dots\} \end{array}$$

$x_{2n} = -1$; $x_{2n-1} = 1$

$K = \{ n ; n = 2k \}$

$b(n) = 2n$

Monotone sequences

$x_1 < x_2 < x_3 < \cdots$ (increasing)

$x_1 > x_2 > x_3 > \cdots$ (decreasing)

$x_1 \leq x_2 \leq x_3 \leq \cdots$ (non-decreasing)

$x_1 \geq x_2 \geq x_3 \geq \cdots$ (non-increasing)

lemma: A monotone sequence is bounded <=> it has a bounded subsequence.

Proof: (⇐) Suppose $x_{n_1} \leq x_{n_2} \leq x_{n_3} \leq \dots \leq b$

since x_{n_k} is a subsequence we can always find $n_k \in \mathbb{N}$ such that $x_n \leq x_{n_k} \leq b \Rightarrow x_n \leq b$.

Ex: 1. Constant sequence

$X_n = a \in \mathbb{R}$; $x_n = \{a, a, a, \dots\}$

2. $X_n = n$

$\{1, 2, 3, 4, 5, \dots\}$

$1 < 2 < 3 < 4 < \dots$

$\mathbb{N} \subseteq \mathbb{R}$

Archimedean

3. $X_n = \frac{1}{n}$ $\quad 0 < \frac{1}{n} \leq 1$

$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

$1 > \frac{1}{2} > \frac{1}{3} > \dots$

4. $X_n = 1 + (-1)^n$; $x_n = \{0, 2, 0, 2, 0, 2 \dots\}$

not monotone

Bounded

$0 \leq X_n \leq 2$

$e = \sum_{n=0}^{\infty} \frac{1}{n!}$

5. $X_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}$ $\quad X_{n+1} > X_n$

$\underbrace{a_1, a_1+s, \dots}_{a_1}$

$2 < x_n < \textcircled{?}\ \textcircled{3}$

$\frac{1}{3.2} < \frac{1}{2.2}$

$1 + \frac{1}{1!} + \frac{1}{2} + \underbrace{\overbrace{\frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{n-1}}}}_{\frac{1}{2}}$

$\frac{1}{4.3.2.1} < \frac{1}{2.2.2.}$

The limit of a sequence

$\lim x_n = a \; := \; \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : n > n_0 \Rightarrow |x_n - a| < \varepsilon$

$n \to +\infty$

by definition $\updownarrow$ $\varepsilon = 0.001$

$|x_n - a| < 0.001$

$x_n \in (a-\varepsilon, a+\varepsilon)$ for infi. many "n"

Remark: We write $\lim x_n$ to indicate $\lim\limits_{n\to+\infty} x_n$.

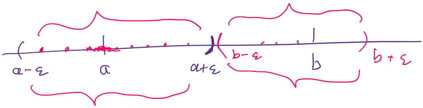
Theorem: If $\lim x_n = a$ and $\lim x_n = b$ then $a = b$.

Proof: Suppose $\lim x_n = a$ and $b \neq a$. Then set $\varepsilon = \frac{|b-a|}{2}$. We can find n_0 such that

$n > n_0 \Rightarrow |x_n - a| < \varepsilon$

$x_n \in (a-\varepsilon, a+\varepsilon)$

$\Rightarrow x_n \notin (b-\varepsilon, b+\varepsilon)$.



So we can't have $\lim x_n = b$ □

Theorem: If $\lim x_n = a$ then for any subsequence

$$\lim_{k\to+\infty} x_{n_k} = a.$$

"If a sequence converges to "$a \in \mathbb{R}$", all the subsequences converge to "a" as well."

Sequence is divergent

not convergent

$n > n_0$; $x_n \in (a-\varepsilon, a+\varepsilon)$

$n_k > n_0$; $x_{n_k} \in (a-\varepsilon, a+\varepsilon)$

Theorem: Every convergent sequence is bounded.

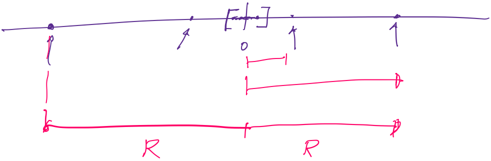
Proof: Take $\varepsilon = 1$ in the definition. $\lim x_n = a$

$\exists n_0 \in \mathbb{N}$; $n > n_0 \Rightarrow x_n \in (a-1, a+1)$

$x_1, x_2, \dots, x_{n_0}$

$R = \max\{|x_1|, |x_2|, \dots, |x_{n_0}|, |a-1|, |a+1|\}$

$x_n \in [-R, R]$



Ex: $x_n = \{0,1,0,1,0,1\}$ is bounded but it's divergent.

Theorem: Every bounded monotone sequence converges

Proof: $x_1 \le x_2 \le x_3 \le \dots \le b$

$\lim x_n = \sup x_n$

$x_1 \ge x_2 \ge \dots$

$\lim x_n = \inf x_n$

Cor: If a monotone sequence x_n has a convergent subsequence then x_n is convergend.

Ex: $x_n = a$; $\lim x_n = a$.

$x_n = n$; $\{1, 2, 3, 4, \dots\}$ it's unbounded

$x_n = \frac{1}{n}$; $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$ $\boxed{\lim x_n = 0}$

$\varepsilon > 0$: $n > n_0 \Rightarrow \left|\frac{1}{n} - 0\right| < \varepsilon$

$\Downarrow$

$\frac{1}{n} < \frac{1}{n_0}$

$0 < \boxed{\frac{1}{n_0} < \varepsilon}$

$\frac{1}{n} < \frac{1}{n_0} < \varepsilon$

$\boxed{\frac{1}{n} < \varepsilon}$

Ex: $0 < a < 1$

$x_n = a^n = \frac{1}{2^n};\ \uparrow \quad \downarrow$

$\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\}$ monotone + bounded $\longrightarrow$ convergent

$0 < x_n < 1$

$\boxed{0}$

Properties of limits

Theorem: $\lim x_n = 0$; y_n is bounded

$$\lim x_n \cdot y_n = 0$$

$|y_n| \leq c$

$|x_n y_n| < \varepsilon$

$|x_n| < \frac{\varepsilon}{\cancel{c}} \cdot \cancel{c} = \varepsilon$

Ex: $\lim_{n\to\infty} \left(\frac{1}{n} \cdot (\sin(n))\right) = 0$

Theorem: $\lim x_n = a$ and $\lim y_n = b$

(i) $\lim x_n \pm y_n = a \pm b$

(ii) $\lim x_n \cdot y_n = a \cdot b$

(iii) $b \neq 0$ $\lim \frac{x_n}{y_n} = \frac{a}{b}$

$2^{\frac{1}{n}} > 2^{\frac{1}{n+1}}$

Ex: $x_n = \sqrt[n]{2}$; $2, 2^{\frac{1}{2}}, 2^{\frac{1}{3}}, 2^{\frac{1}{4}}, \dots$

$\lim \sqrt[n]{2} = L$ $(=1)$ $0 < x_n \leq 2$

Consider the subsequence $y_n = \underbrace{x_{n(n+1)}}$

$$L = \lim y_n = \lim \sqrt[n(n+1)]{2} = \lim 2^{\frac{1}{n(n+1)}} = \lim 2^{\frac{1}{n} - \frac{1}{n+1}}$$

$$L = \frac{\lim 2^{\frac{1}{n}}}{\lim 2^{\frac{1}{n+1}}} = \frac{L}{L} = \boxed{1}$$

$\boxed{L = 1}$

$L = 3L + 4$

Theorem: If $\lim x_n = a$ and $a > 0$ then $\exists n_0 \in \mathbb{N}$: $n > n_0 \Rightarrow x_n > 0$.

$\liminf x_n$

$\limsup x_n$