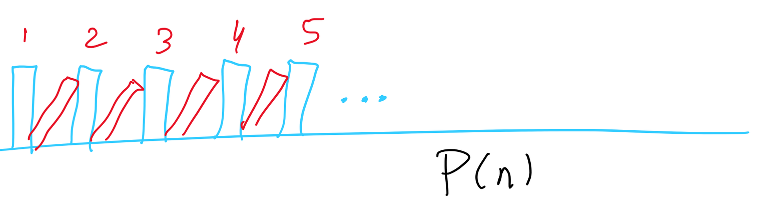
Wednesday, September 25, 2024 12:35 PM

Axiom: If $X \subseteq \mathbb{N}$, and $1 \in X$,
$n \in X \Rightarrow n+1 \in X$
Then, $X = \mathbb{N}$.

$X = \{x;\ P(x) \text{ is true}\} = \mathbb{N}$

"If a certain statement about the natural numbers, say $P(n)$, has the following properties.
$P(1)$ is true $\Leftrightarrow 1 \in X$
$P(n)$ true $\Rightarrow P(n+1)$ true $n \in X \Rightarrow n+1 \in X$
Then $P(n)$ is true $\forall n \in \mathbb{N}$."
↳ For all



$P(n)$

Proposition: $1 + 2 + \dots + n = \dfrac{n \cdot (n+1)}{2}$, $n \in \mathbb{N}$

$P(1)$ $1 = \dfrac{1 \cdot (1+1)}{2}$ ✓ true

$P(n)$ is true $\overset{?}{\Longrightarrow} P(n+1)$

$1+2+\dots+n = \dfrac{n \cdot (n+1)}{2} \Rightarrow 1+2+\dots+n+1 = \dfrac{n \cdot (n+1)}{2} + n+1$

$= \dfrac{n(n+1) + 2(n+1)}{2}$

$1+2+\dots+n+1 = \dfrac{(n+1)(n+2)}{2}$

$P(n+1)$ is true

$n! = 1.2.3.4.5.\dots.n$

Proposition: $\overbrace{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1) = \dfrac{(2n)!}{2^n n!}}^{P(n)}$

Proof: $n=1$
$1 = \dfrac{2!}{2.1} = 1$

$n=2$
$1.3 = \dfrac{(2.2)!}{2^2 . 2!} = \dfrac{4!}{4.2} = \dfrac{4.3.2.1}{4.2} = 3$

$P(n)$ is true $\Rightarrow 1.3.5\dots(2n-1) = \dfrac{(2n)!}{2^n . n!}$
?
$P(n+1)$

$1.3.5\dots(2n+1) = \dfrac{(2n+2)!}{2^{n+1} . (n+1)!} = \dfrac{(2n+2)(2n+1)!}{2.2^n . (n+1) . n!} = \dfrac{(2n+1)!}{2^n n!}$

$1.3.5\dots(2n-1).(2n+1) = \dfrac{(2n)!}{2^n . n!} . (2n+1)$

$= \dfrac{(2n+1)!}{2^n . n!}$

$P(n+1)$ is true

(4.1) $1 + 3 + 5 + \dots + (2n-1) = n^2$

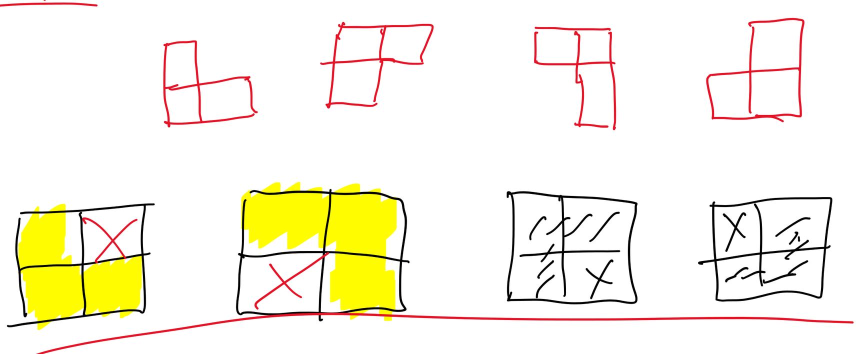
$P(1)$ $1 = 1^2 = 1$ true

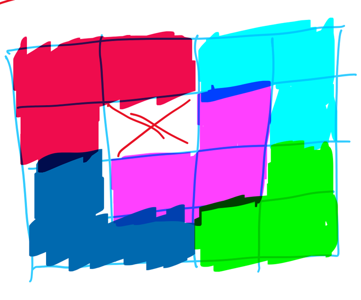
$P(n)$ is true $1+3+\dots+(2n-1) = n^2$

$P(n+1)$ is true
$\underbrace{1+3+5+\dots+2n-1}_{n^2} + 2n+1 = n^2 + 2n + 1$
$= (n+1)^2$

∴ By induction, $P(n)$ is always true.

Example 4.

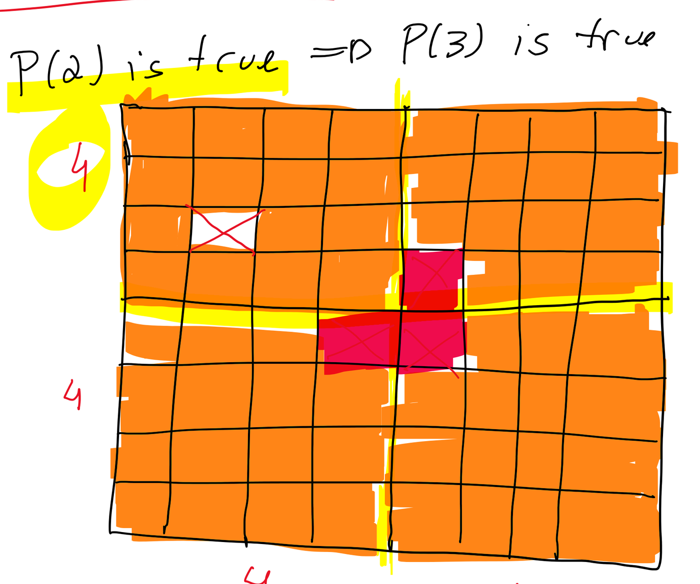




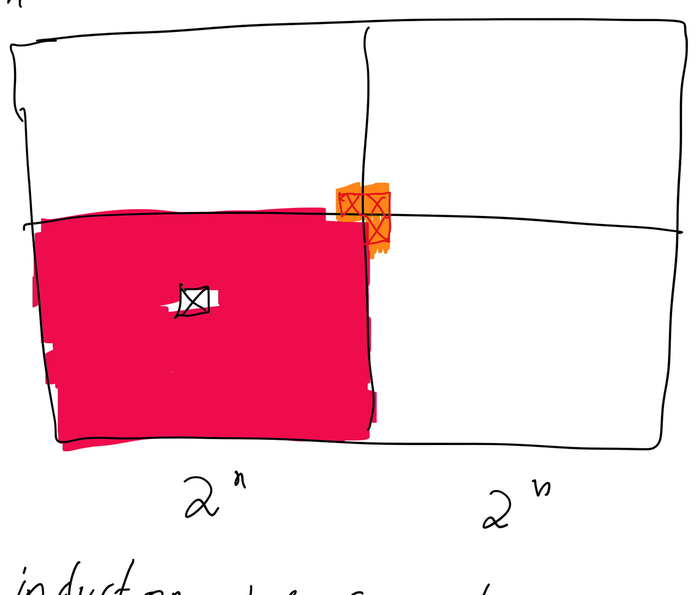
Conjecture We can always cover a $2^n \times 2^n$ chessboard with "L" shapes, if we remove one square.

$P(1)$ is true

$P(2)$ is true $\Rightarrow P(3)$ is true



$P(n)$ is true $\Rightarrow P(n+1)$ is true



∴ By induction we can always cover a $2^n \times 2^n$ chessboard if we remove one square.

Strong induction

$P(1)$ is true

$P(1), P(2), \dots, P(n)$ is true $\Rightarrow P(n+1)$ is true

∴ $P(n)$ is always true.

Theorem: If $n \geq 2$, then n is a (unique) product of primes.

$5.2.2$
$2.2.5 = p_1 . p_2 . p_3$

$p_1 p_2 p_3 \dots p_n = q_1 . q_2 \dots q_n$
$p_i = q_i$

$2, 3, 4 = 2.2, 5, 20 = 2.2.5$
$30 = 2.3.5$

Proof: $P(2)$, $2 = 2$ and 2 is prime.

Assume $P(2), P(3), \dots, P(n)$.
$P(n+1)$
$n+1$ is either prime or not prime.
Suppose $n+1$ is not prime, then:
$n+1 = p \cdot q$, $p, q < n+1$

Using the induction hypothesis on "p", "q"
"p" is a product of primes, "q" is a product of primes then
$n+1 = p \cdot q$ is also a product of primes

Ex: Chocolate bar