Wednesday, September 25, 2024 3:37 PM

Intervals

K ordered field. $a, b \in K$

$a < b$

$[a,b] = \{x \in K ; a \leq x \leq b\}$ "closed interval"

$(a,b) = \{x \in K ; a < x < b\}$ "open interval"

$[a,b) =$ " $a \leq x < b$ "half-open"

$(a,b]$ " $a < x \leq b$

$(-\infty, b]$ " $x \leq b$

$(a, +\infty)$ " $x > a$

$x > 0 \Rightarrow x \in P$

$(-\infty, \infty) = K$

(a,a) or $[a,a]$ is called degenerate.

if $a \neq b$ and $(a,b), [a,b), \dots$ is non-degenerate.

$$|x| = \max\{x, -x\} = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Theorem: Take $x, y \in K$. The following are equivalent:

(i) $-y \leq x \leq y$

(ii) $x \leq y$ and $-x \leq y$

(iii) $|x| \leq y$

"$\leq$" $\Leftrightarrow$ "$<$"

Proof: (i) $\Rightarrow$ (ii) $x \leq y$; $-y \leq x \Rightarrow -x \leq y$

(ii) = (iii)

(iii) $\Rightarrow$ (i)

Corollary: Let $x, a, \varepsilon \in K$ then:

$$|x - a| \leq \varepsilon \Leftrightarrow a - \varepsilon \leq x \leq a + \varepsilon$$

Theorem:

(i) $|x + y| \leq |x| + |y|$

(ii) $|x \cdot y| = |x| \cdot |y|$

(iii) $|x| - |y| \leq \big| |x| - |y| \big| \leq |x - y|$

(iv) $|x-z| \leq |x-y| + |y-z|$

Let K be an ordered field and $X \subseteq K$.

An upper bound of X is $M \in K$ such that

$$a \leq M \quad \forall a \in X$$

A lower bound of X is $m \in K$ such that

$$m \leq a, \quad \forall a \in X$$

A set $X \subseteq K$ is bounded from above if it has an upper bound, bounded from below if it has a lower bound. $X \subseteq K$ is bounded if it has both upper and lower bounds.

Ex: The well-ordering principle says that any $X \subseteq \mathbb{N} \subseteq \mathbb{Q}$ is bounded from below. The natural numbers $\mathbb{N}$, are not bounded from above inside $\mathbb{Q}$.

Ex: $\mathbb{Q}(t)$ $\quad P = \{ \frac{p(t)}{q(t)} \mid \overset{\text{the lead coeff.}}{p(t) \cdot q(t)} \in \mathbb{N} \}$

$n \in \mathbb{N}$

$r(t) - n > 0$

$r(t) = t$

$1 \cdot t - n > 0$

The rational function $r(t) = t$ is an upper bound for $n \in \mathbb{N}$. Hence, $\mathbb{N}$ is bounded from above in $\mathbb{Q}(t)$.

Theorem: Let K be an ordered field. The following are equivalent:

(i) $\mathbb{N}$ is not bounded from above;

(ii) Given $a, b \in K$, with $a > 0$, $\exists n \in \mathbb{N}$ such that $na > b$

(iii) Given $a > 0$, $\exists n \in \mathbb{N}$ such that $0 < \frac{1}{n} < a$.

A field K satisfying one (and hence all) of the above is called Archimedean field.

Proof: (i) $\Rightarrow$ (ii) $\frac{b}{a} \in K$, $\exists n$ such that $n > \frac{b}{a} \Leftrightarrow \underline{na > b}$

(ii) $\Rightarrow$ (iii) $b = 1$ in (ii) $a > 0, \exists n$ $n \cdot a > 1 \Leftrightarrow a > \frac{1}{n}$

(iii) $\Rightarrow$ (i) $\frac{1}{n} < \frac{1}{a} \Leftrightarrow \underline{\underline{a < n}}$

The Real numbers $\mathbb{R}$

Let K be an ordered field and $X \subseteq K$ bounded from above. The supremum $(\sup X)$ of X as the least upper bound.

$\sup X \in K$ has the following properties:

(i) (upper bound) $\forall x \in X$, $x \leq \sup X$.

(ii) (least upper bound) If $c \in K$ such that $x \leq c$ $\forall x \in X$ then $\sup X \leq c$. (If $c \in K$ such that $c < \sup X$ then $\exists x \in X$: $c \leq x$.)

Lemma: If $\sup X$ exists, it is unique.

$a = \sup X$, $b = \sup X$

$a \leq b$

$b \leq a$

$\therefore \underline{\underline{a = b}}$

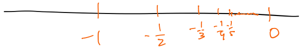
lemma: If X has a maximum element M then $\sup X = M$.

Ex: $I_n = \{1, 2, 3, \dots, n\}$

$\sup I_n = \max I_n = n$.

Ex: $X = \{-\frac{1}{n} ; n \in \mathbb{N}\} = \{-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \dots\}$

$\sup X = 0$



0 is clearly an upper bound for X

$$-\frac{1}{n} < 0$$

Suppose $c < 0$, we claim $\exists n \in \mathbb{N}$ such that

$c < -\frac{1}{n}$ (follows from the fact that $\mathbb{Q}$ is Archimedean)

$X \subseteq K$ ord. field

X is bounded from below.

infimum of X ($\inf X$):

:= greatest lower bound

$\inf X$ has the following properties

(i) (lower bound) $\inf X \leq x, \forall x \in X$

(ii) (greatest lower bound) If $c \leq x, \forall x \in X$ then $c \leq \inf X$.

Remark: $\inf X$ is unique; if X has a minimum then $\inf X = \min X$

lemma: Every finite set has a supremum and infimum

Ex: $X = (a, b) \subseteq K$

$$\inf X = a$$

$$\sup X = b$$

a is obviously a lower bound we claim $\underline{\inf X = a}$
for instance, if $c > a$, $x = \frac{a+c}{2}$ then "c" is not a lower bound since $x \leq c$.

lemma 89. There is no $x \in \mathbb{Q}$ such that $x^2 = 2$.

Proof: Suppose $\frac{p}{q} \in \mathbb{Q}$ and $\left(\frac{p}{q}\right)^2 = 2$

$$\underline{p^2} = 2q^2$$

The number of factors equal to "2" in p^2 is even, same thing happens with q^2 as well, but then $2.q^2$ has an odd amount of factors "2" so $p^2 \neq 2q^2$

Proposition: $X = \{x \in \mathbb{Q} ; \underline{x \geq 0} \text{ and } \underline{x^2 < 2}\}$
$Y = \{y \in \mathbb{Q} ; \underline{y \geq 0} \text{ and } \underline{y^2 > 2}\}$

There are no rational numbers a, b such that
$a = \sup X$, $b = \inf Y$.

Proof: First we claim that X doesn't have a maximum element. We have to prove that given any $x \in X$, $\underline{x+r \in X}$ for some r.

Take $r<1$ satisfying $0<r<\frac{2-x^2}{2x+1}$, we have

$r^2<r$

$(x+r)^2 = x^2+2rx+r^2$

$< x^2+2rx+r = x^2+r(2x+1)$

$< \cancel{x^2}+2-\cancel{x^2}$

$=2$

$\therefore (x+r)^2<2$, i.e. $x+r \in X$.

Similarly, Y doesn't have a minimum.

$\underline{x \in X,\ y \in Y \Rightarrow x<y}$ ($x^2<2<y^2$

$x^2<y^2$

$0<y^2-x^2=(y-x)(y+x)$

$\therefore\ 0<y-x$

$x<y$)

Suppose $a=\sup X$,

$a \neq \max X$, $a \neq \min Y$

$a^2<2$ is false

$2<a^2$ is false

The only possibility would be $a^2=2$, but by Pythagoras that is impossible.

We say an ordered field K is complete if every set $X \subseteq K$ bounded from above has a supremum.