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Corollary: A set X is infinite $\iff \exists f: X \to Y$ bijection, $Y \subsetneq X$ proper subset.

Proof ($\Rightarrow$) If a set is infinite, there is a countable subset $Y \subseteq X$; suppose $Y = \{x_1, x_2, x_3, \dots\}$

Let $Z := (X - Y) \cup \{x_2, x_4, x_6, \dots\}$, then:

$$f: X \to Z$$
$$x \mapsto x \text{ if } x \notin Y$$
$$x_i \mapsto x_{2i}$$

is a bijection (Notice that Z is proper).

($\Leftarrow$) Proved already.

Theorem: Every subset $X \subseteq \mathbb{N}$ is countable.

Proof: If X is finite then it is countable, so suppose X infinite. Set $a_1 = \min X$, $X_1 = X$; put $f(1) = a_1$. Set $X_2 = X - a_1$, $a_2 = \min X_2, \dots$ proceed by induction, then $a_n = \min X_n$, $X_n = X - \{a_1, a_2, \dots, a_{n-1}\}$. The correspondence $f: \mathbb{N} \to X$ is a bijection. It is injective by construction. Suppose it's not surjective. Then $\exists x \in X$ such that $x \notin f(\mathbb{N})$, $x \in X_n$ $\forall n \in \mathbb{N}$, hence $f(n) < x$ $\forall n \in \mathbb{N} \Rightarrow f(\mathbb{N})$ is bounded $\Rightarrow f(\mathbb{N})$ is finite $\lightning$ $\square$

Corollary: Let X be countable. $Y \subseteq X$ then Y is countable.

Proof: Suppose X not finite, $f: \mathbb{N} \to X$.

$$f: f^{-1}(\mathbb{N}) \xrightarrow{\simeq} Y$$
$$f^{-1}(\mathbb{N}) \subseteq \mathbb{N} \cong \mathbb{N}$$

Corollary: The set of all primes is countable.

Proof: Notice that $\{x; x \text{ is prime}\} \subseteq \mathbb{N}$, hence countable.

Corollary: Let Y countable, $f: X \to Y$ injective, then X is countable.

Proof: Notice that $f(X) \subseteq Y$, $f(X)$ is countable.

$$f: X \xrightarrow{\simeq} f(X) \subseteq Y$$
$$X \cong \mathbb{N} \cong f(X)$$

Example: $\mathbb{Z}$ is countable.

$$f: \mathbb{Z} \to \mathbb{N}$$
$$0 \mapsto 1$$
$$n \mapsto 2n, \ n > 0$$
$$n \mapsto -2n+1, \ n < 0$$

Corollary: Let X countable, $f: X \to Y$ surjective, then Y is countable.

$$f^{-1}: Y \to X, \text{ inj.}$$

Example: $\mathbb{N} \times \mathbb{N}$ is countable.

$$f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$
$$(m, n) \mapsto 2^m \cdot 3^n$$

Corollary: X, Y are countable then $X \times Y$ is countable

$X \cup Y$ "

$\bigcup_{i=1}^{\infty} X_i$ "

Example: $\mathbb{Q}$ is countable.

$$f: \mathbb{Z} \times \mathbb{Z}^* \to \mathbb{Q}$$
$$(m, n) \to \frac{m}{n}$$

$\mathbb{Q}^*, \mathbb{Z}^*, \mathbb{R}^*$

Uncountable sets

We say that X, Y have the same cardinality if there is a bijection $f: X \to Y$.

$$\text{card}(X) = \text{card}(Y)$$

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Naive Set theory

Remark: If X is finite then $\text{card}(X) = |X|$.

$$\text{card}(X) < \text{card}(Y)$$

If there is an injective function $f: X \to Y$ that is never surjective.

$$\text{card}(\mathbb{N}) = \aleph_0 \ (\text{Aleph zero})$$

Notice that for any X infinite:

$$\text{card}(\mathbb{N}) \le \text{card}(X)$$

Theorem (Cantor's theorem) Let X, Y be sets such that $|Y| \ge 2$. Then there is no surjective function

$$\phi: \underset{\mathbb{N}}{X} \to \underset{F(\mathbb{N}; Y) = P(\mathbb{N})}{\mathcal{F}(X; Y)}$$

Proof: (Diagonal's method): Let $\phi: X \to \mathcal{F}(X, Y)$ be a function, we claim that ϕ is not surjective. Let's denote $\phi(a) = \phi_a \in \mathcal{F}(X, Y)$. Given $a \in X$, consider $f(a) \neq \phi_a(a)$. Then that implies that $f \neq \phi_a$ for any $a \in X$. $\square$

Example: $\mathcal{F}(\mathbb{N}; \mathbb{N})$ is not countable.

There is no bijection $f: \mathbb{N} \to \mathcal{F}(\mathbb{N}, \mathbb{N})$.

Example: $\mathbb{R}$ is not countable.

1. $a, b \in \mathbb{N}$, $\exists m \in \mathbb{N}$ $ma > b$.

   Proof: Suppose $ma \le b$ $\forall m \in \mathbb{N}$. Consider the set $X = \{ma; m \in \mathbb{N}\}$. Then the set X is bounded, it has to be finite $\lightning$, a contradiction.

2. $a \in \mathbb{N}$. Consider X: $X \subseteq \mathbb{N}$

   $\boxed{a \in X}$; $\boxed{n \in X \Rightarrow n+1 \in X}$

   $\vdash$ X contains all natural numbers greater than or equal to "a".

   Proof: By induction, $a+1 \in X$, $a+n \in X$ $\forall n \in \mathbb{N}$.

3. $a < b$, $\nexists c \in \mathbb{N}$; $a < c < b$

   $\vdash \forall n \in \mathbb{N}, n \neq 1$; n has a predecessor.

   a pred. b

   $b = s(a)$

   axiom 2: $\mathbb{N} - s(\mathbb{N}) = \{1\}$, $s: \mathbb{N} \to \mathbb{N}$

4. $(1 + \dots + n) = \dfrac{n \cdot (n+1)}{2}$

   $X = \{n \in \mathbb{N}; 2(1 + \dots + n) = n \cdot (n+1)\}$

   $1 \in X$ ✓

   $n \in X$; $2(1 + \dots + n) = n \cdot (n+1)$

   $n+1 \in X$? $2(1 + \dots + n + n+1) = 2(1 + \dots + n) + 2(n+1)$
   $= n \cdot (n+1) + 2(n+1)$
   $= (n+1)(n+2)$ $\square$

   $\therefore X = \mathbb{N}$

c. $n \ge 4 \Rightarrow n! > 2^n$

   4: $4! > 2^4 \iff 24 > 16$ ✓

   $(n+1)! > 2^{n+1}$? $(n+1) > 2$

   $(n+1)! = (n+1) n! > (n+1) 2^n$
   $> 2 \cdot 2^n$
   $> 2^{n+1}$ $\square$

10. "A is countable $\not\Rightarrow$ $P(A)$ is countable"

   $P(A) \cong F(A; \{0,1\})$

   $\mathbb{N} \to F(X; Y)$

   $B \subseteq A$

   $f_B: A \to \{0,1\}$

   $f_B(x) = 1$, $x \in B$

   $f_B(x) = 0$, $x \notin B$

"A is countably infinite then $P(A)$ is uncountable"

$$\mathbb{N} \to P(A) \approx \mathcal{F}(A; \{0,1\})$$

Cantor's theorem: There is no bijection between $\mathbb{N}$ and $P(\mathbb{N})$

$$\text{card}(\mathbb{N}) < \text{card}(P(\mathbb{N}))$$