Monday, September 16, 2024 3:54 PM

13. Start with

$$\mathbb{N} = X \cup Y$$

where $X = \{n;\ n \text{ even}\}$; $Y = \{n;\ n \text{ is odd}\}$

$$X \cap Y = \emptyset$$

define $X_2 = \{n;\ n = 4\cdot k;\ k \in \mathbb{N}\}$, $X_2 \subseteq X$

$$X = X_2 \cup X_2^c$$

$$\mathbb{N} = X_2 \cup X_2^c \cup Y$$

$X_3 = \{n;\ n = 8\cdot k\}$, $X_3 \subseteq X_2$

$$X_2 = X_3 \cup X_3^c$$

$$\mathbb{N} = X_3 \cup X_3^c \cup X_2^c \cup Y$$

By induction:

$$\mathbb{N} = Y \cup X_2^c \cup X_3^c \cup X_4^c \cdots$$

$$X_j = \{n;\ n = 2^j \cdot k\}$$

8. $f: \mathbb{N} \to \mathbb{N}$ surj., $f^{-1}(n)$ is infinite

$\pi: \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$

$(x, y) \longmapsto x$

$\mathbb{N}$ b

$\pi^{-1}(n)$ is infinite

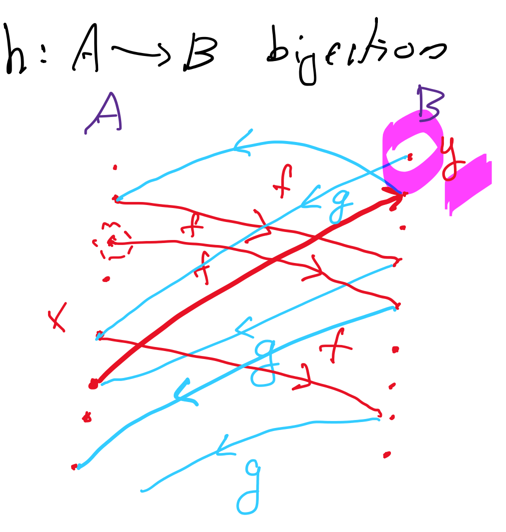
$\pi^{-1}(1) = \{(1, y);\ y \in \mathbb{N}\}$

$f = \pi \circ b : \mathbb{N} \longrightarrow \mathbb{N}$

$f^{-1}(n)$ is infinite.

16. $f: A \longrightarrow B$, $g: B \longrightarrow A$ injectives.

$\vdash h: A \to B$ bijection



$h(x) = f(x)$

$h(x) = g^{-1}(x)$

$h(x) = f(x)$

8. $f: A \longrightarrow B$;

$X \subseteq A$

$f(A - X) = f(A) - f(X)$

f is injective $\Leftrightarrow$ $f(X^c) = f(X)^c$

$(\Rightarrow)$ $f(X^c) \subseteq f(X)^c$ and $f(X)^c \subseteq f(X^c)$

$y \in f(X^c)$ $\qquad$ $y \in f(X)^c$

$y = f(x)$; $x \notin X$ $\qquad$ $y \notin f(X)$

$\Downarrow$ $\qquad$ $y \neq f(x)$ for any $x \in X$

$y \notin f(X)$ $\qquad$ $y \in f(X^c)$

$(\Leftarrow)$ $f(A - X) = f(A) - f(X)$

$\vdash f$ is injective.

$f(x) = f(y) \Leftrightarrow x = y$

$x \neq y \Leftrightarrow f(x) \neq f(y)$

$X = \{a\}$

$b \neq a$

$f(A - \{a\}) = f(A) - f(a)$

$f(b) \in f(A - \{a\}) = f(A) - f(a)$

$f(b) \in f(A) - f(a)$

$f(b) \neq f(a)$

5. $n+1$ < $n+1$ is prime ✓ / $n+1$ is composite

Existence

$n+1 = x \cdot y$, $x, y < n+1$

Since $x, y < n+1 \Rightarrow x, y$ have a prime dec.

Uniqueness: $n = p_1 \cdot p_2 \cdots p_n = q_1 \cdot q_2 \cdots q_m$

$p \mid a \cdot b$; a, b primes
$a = p$ or $b = p$

$n = m$ $\qquad$ $P_i = q_i$

Suppose $n = p_1 \cdot p_2 \cdots p_n = q_1 \cdots q_m$ are two prime decomposition of the same number "n".

Notice that $p_1 \mid n \Rightarrow p_1 \mid q_1 \cdots q_m$

$\Rightarrow q_1 = p_1$

$p_2 \mid n \Rightarrow p_2 \mid q_2 q_3 \cdots q_m \Rightarrow q_2 = p_2$

If we keep doing this procedure we find that

$P_i = q_i$

up to re-labeling p_i' and q_i's.

14. $X \subseteq \mathbb{N}$ infinite $\qquad$ $x < y \Rightarrow f(x) < f(y)$

$f: \mathbb{N} \to X$ (inc), bij., unique.

$f(1) = \min X$ $\qquad$ $f(1) = a \in X$

$X_2 = X - f(1)$ $\qquad$ $f(2) = b \in X - \{a\}$

$f(2) = \min X_2$ $\qquad$ $f(1) < f(2)$

$f(1) < f(2)$ $\qquad$ $a < f(2)$

$a < b$

$X_3 = X - f(1) - f(2)$

$f(3) = \min X_3$

$f(1) < f(2) < f(3)$

By induction,

$f(n) = \min X_n$

$X_n = X - \{f(1), f(2), \ldots, f(n-1)\}$

$f(1) < f(2) < \cdots < f(n) < f(n+1) < \cdots$