Wednesday, September 11, 2024 12:30 PM

Sets

A set X is a collection of objects.

$$X = \{x_1, x_2, x_3, \ldots\}$$

Remark: The collection has to be well defined.

Ex: $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ $2 \in \mathbb{N}$, $1 \in \mathbb{N}$, $\frac{1}{2} \notin \mathbb{N}$

$P = \{2, 3, 5, 7, 11, 13, \ldots\}$ $10 \notin P$, $101 \in P$

$X = \{\{1\}, \{2\}, \{1,2\}, \emptyset\}$ $\{1\} \in X$

$\emptyset$ = Empty set := A set with no element.

Notation:

$x \in X$

"x is an element of X"

$x \notin X$

"x isn't an element of X"

$A \subseteq B$ if every element of "A" belongs to "B"

Ex: $\mathbb{N} \subseteq \mathbb{Q}$ "$\mathbb{N}$ is a subset of $\mathbb{Q}$"

$P \subseteq \mathbb{N}$

$\{x; x \text{ is a square}\} \subseteq \{x; x \text{ is a rectangle}\}$

Notation:

$X = \{x; x \text{ has some property}\}$

"The set of all x such that ..."

$X = \{x; x \text{ is a triangle}\}$

$Z = \{x; x \text{ is a school subject}\}$

Y ("Not a set!") $= \{x; x \text{ is a tall person}\}$

$X = \{x; x \text{ is a hot city}\}$

$X = \{x; x \text{ is a dog}\}$

$X = \{x; x \text{ is a US-state}\}$

...

$X \subseteq X$

Banach-Tarski paradox

Naive Set theory, P. Halmos

Proper subset: $X \subseteq Y$ but $X \neq Y$.

$\{1, 2\}$ is a proper subset of $\{1, 2, 3\}$

Notation: $X \subsetneq Y$ $\mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\}$

Set builder Notation

Odd numbers $= \{x \in \mathbb{N} \mid x = 2k+1;\ k \in \mathbb{N}\}$

$X = \{x \in \mathbb{Z};\ x = 10 \cdot k,\ k \in \mathbb{Z}\}$

$\mathbb{Q} = \{x;\ x = \frac{p}{q};\ p, q \in \mathbb{Z} \text{ and } q \neq 0\}$

Proposition: $\underbrace{\{n \in \mathbb{Z};\ 6 \mid n\}}_{A} \subseteq \underbrace{\{n \in \mathbb{Z};\ 3 \mid n\}}_{B}$

$A \subseteq B$

Idea of Proof: Take $a \in A$ and conclude that $a \in B$.

Take $n \in A$, then $n = 6 \cdot k \Rightarrow n = 3 \cdot 2 \cdot k$

$n = 3 \cdot p$

$3 \mid n \Rightarrow n \in B$.

Conversely, if we want to show that $A = B$ then we must show that $B \subseteq A$ as well!

3.24. $\{5k+1;\ k \in \mathbb{Z}\} = \{5k+6;\ k \in \mathbb{Z}\}$

$A \subseteq B$, $B \subseteq A \Rightarrow A = B$

$n \in A \Leftrightarrow n = 5k+1$ $(= 5p+6)$

$\Updownarrow$

$n = 5(k-1) + 6$

$\Updownarrow$

$n \in B$

$5k+1 = 5p+6$

$5(k-p) = 5$

$k - p = 1$

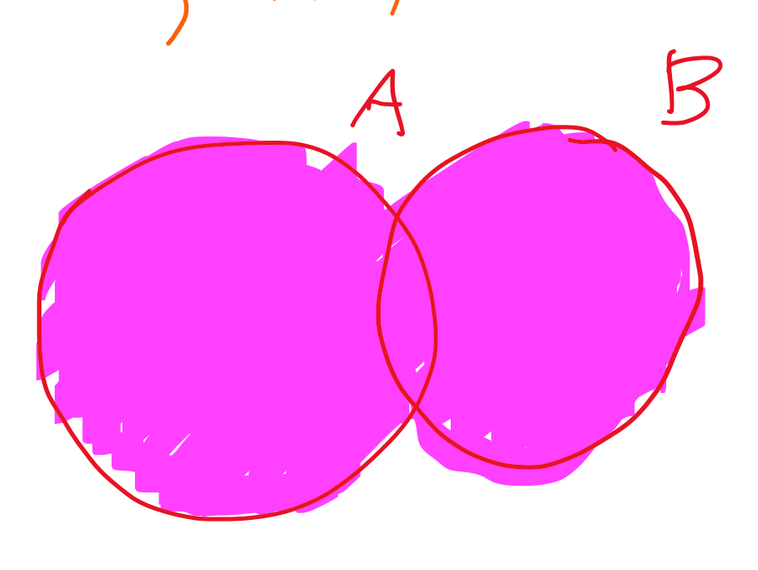
$p = k-1$

Creating new sets out of "old sets"

Union

Given two sets A and B:

$A \cup B = \{x;\ x \in A \text{ or } x \in B\}$



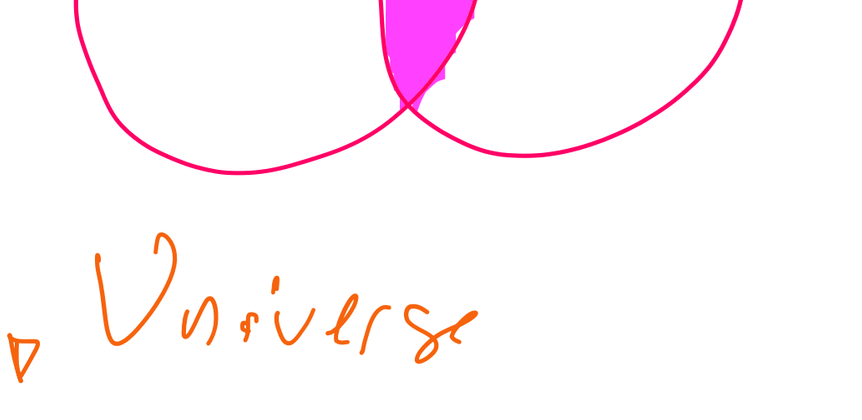
Intersection

$A \cap B = \{x;\ x \in A \text{ and } x \in B\}$

Complement

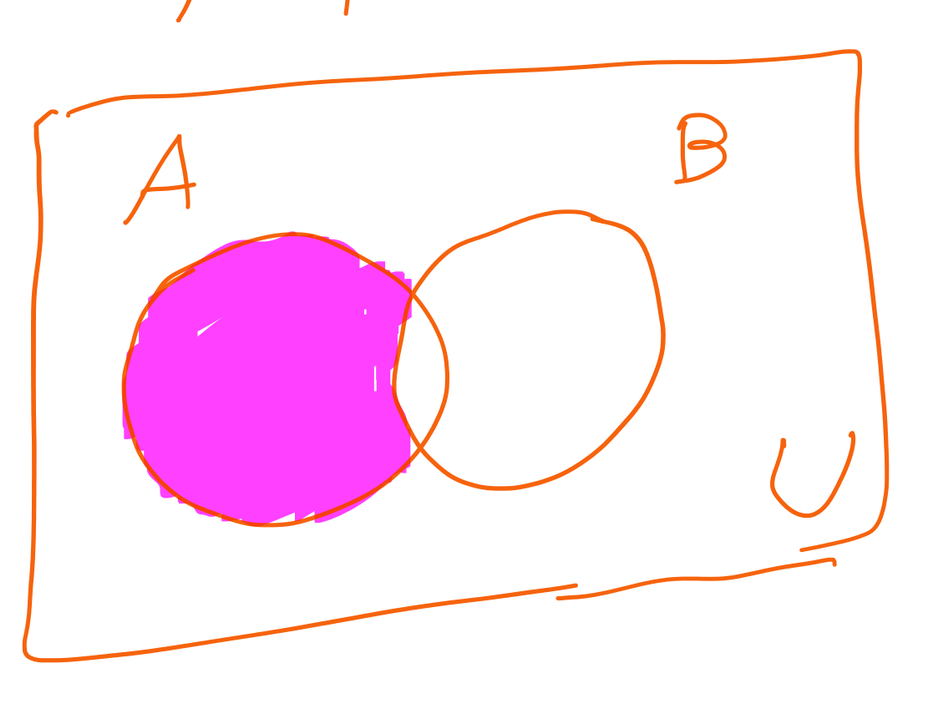
$A \subseteq U$ → Universe

$A^c = \{x \in U;\ x \notin A\}$



Set difference

$A - B = \{x;\ x \in A \text{ and } x \notin B\} = A \cap B^c$



Power set

$\mathcal{P}(A) = \{B;\ B \subseteq A\}$

Ex: $A = \{1, 2\}$, $|A| = 2$

$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, $|\mathcal{P}(A)| = 4$

Cardinality

$|A| = n$ if A has "n" elements.

$|\mathbb{N}| = \text{Aleph } 0$ $f: X \to Y$ bijection

$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| \neq |\mathbb{R}|$

$|\mathbb{N}| < |\mathbb{R}|$

The cartesian Product

$A \times B = \{(a, b);\ a \in A \text{ and } b \in B\}$

ordered pair $= (a, b)$

$(a, b) = (c, d) \Leftrightarrow a = c,\ b = d$

$\mathbb{N} \times \mathbb{N} = \{(n, m);\ n, m \in \mathbb{N}\}$

$(1, 100) \in \mathbb{N} \times \mathbb{N}$

$-1 \notin \mathbb{N} \times \mathbb{N}$

Remark: Given A, B sets.

$P(A) \subseteq P(B) \Rightarrow A \subseteq B$.

$P(A)$ is the set of all subsets of A, i.e. $X \subseteq A$.

In particular, $A \in P(A) \Rightarrow A \in P(B) \Rightarrow A \subseteq B$.

Theorem: (De morgan's laws)

1. $(A \cup B)^c = A^c \cap B^c$
2. $(A \cap B)^c = A^c \cup B^c$

Proof: (1). $(A \cup B)^c \subseteq A^c \cap B^c$ ($\supseteq$)

Take $x \in (A \cup B)^c \Leftrightarrow x \notin A \cup B$

$\Updownarrow$

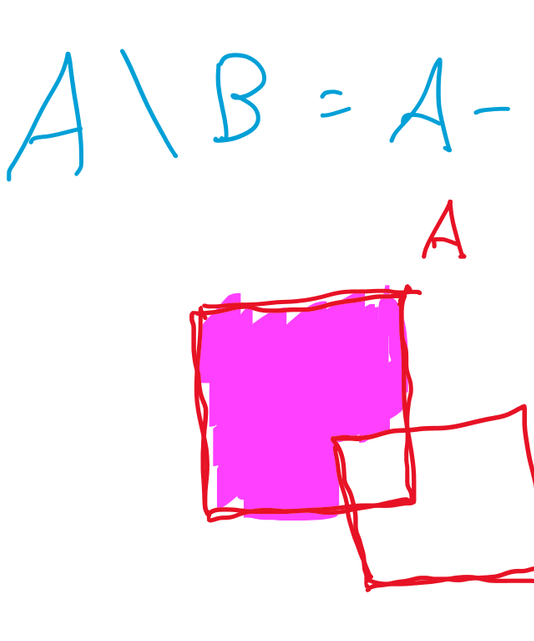
$x \notin A$ and $x \notin B$

$\Updownarrow$

$x \in A^c \cap B^c$ □

3.2 $A \setminus B = A - B$

$P(A)$, $\emptyset$



$|A|$ = The number of "things" inside the box.

3.3

(a) $\{n \in \mathbb{Z};\ -5 \leq n < 4\} = \{-5, -4, -3, -2, 1, 0, \ldots, 3\}$

(c) $\{3n;\ n \in \mathbb{Z} \text{ and } |2n| < 8\} = \{-9, -6, -3, 0, 3, 6, 9\}$

(e) $\emptyset \times \{1, 2, 3\} = \emptyset$

$\{(a, b);\ a \in \emptyset \text{ and } b \in \{1, 2, 3\}\}$

Arithmetic Progression

(f) $\{5n + 3;\ n \in \mathbb{Z}\} = \{\ldots, -7, -2, 3, 8, 13, 18, \ldots\}$

3.4 $A = \{1, 2, 3, 4, 5\}$; $B = \{3, 4, 5, 6, 7\}$, $C = \{1, 3, 5, 7\}$

$U = \{1, \ldots, 8\}$ $A, B, C \subseteq U$

(a) $A \cup B = \{1, \ldots, 7\} = \{8\}^c = U - \{8\}$

(c) $A \setminus C = \{2, 4\}$

(f) $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$

$\{8, 6, 7\} \cap \{1, 2, 8\} \cap \{2, 4, 6, 8\}$

$\{8\}$