Monday, October 14, 2024 3:34 PM

Ex: $\sum \frac{1}{n(n+1)} = 1$ $\qquad \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$s_1 = \frac{1}{1\cdot 2} = 1 - \cancel{\frac{1}{2}}$

$s_2 = \frac{1}{2\cdot 3} = \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$

$s_3 = \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}$

$\vdots$

$\boxed{s_n = 1 - \cancel{\frac{1}{n+1}}}$ $\qquad \lim s_n = 1$

Ex: $\sum (-1)^n = -1 + 1 - 1 + 1 - \dots$

is divergent

since $\lim (-1)^n \neq 0$.

$\lim (-1)^n$

$\{-1, 1, -1, 1, \dots\}$

$x_{2n} = 1$

$x_{2n-1} = -1$

Theorem: Let $a_n \geq 0$. $\sum a_n$ converges $\Leftrightarrow$ s_n is bounded.

$\Downarrow$

$\lim s_n < \infty$

$\hookrightarrow a_1 + a_2 + \dots + a_n$

$s_1 = a_1$

$\wedge$

$s_2 = a_1 + a_2$

$\wedge$

$s_3 = s_2 + a_3$

$\wedge$

s_n montone + bounded

$\Downarrow$

conv.

Corollary: (Comparison principle) $\sum a_n$, $\sum b_n$ such that $a_n, b_n \geq 0$. If $\exists n_0 \in \mathbb{N}$, $n > n_0 \Rightarrow \boxed{a_n \leq c\, b_n}$ for some $c \in \mathbb{R}$ then if b_n converges $\Rightarrow$ a_n converges

If a_n diverges then b_n diverges.

Ex: $r > 1$, $\sum \frac{1}{n^r}$ converges. $\qquad s_n = 1 + \frac{1}{2^r} + \dots + \frac{1}{n^r}$ is bounded.

$$1 + \frac{1}{2^r} + \frac{1}{3^r} + \frac{1}{4^r} + \dots$$

$$s_{2^n-1} = 1 + \frac{1}{2^r} + \dots + \frac{1}{(2^n-1)^r}$$

$$= 1 + \left(\frac{1}{2^r} + \frac{1}{3^r}\right) + \left(\frac{1}{4^r} + \dots + \frac{1}{7^r}\right) + \dots + \frac{1}{(2^n-1)^r}$$

$$< 1 + \left(\frac{1}{2^r} + \frac{1}{2^r}\right) + \left(\frac{1}{4^r} + \frac{1}{4^r} + \dots + \frac{1}{4^r}\right) + \dots$$

$$= 1 + \frac{2}{2^r} + \frac{4}{4^r} + \dots + \frac{2^{n-1}}{2^{(n-1)r}}$$

$$= \sum_{j=0}^{n-1} \left(\frac{2}{2^r}\right)^j \quad \text{since } r > 1 \quad \frac{2}{2^r} < 1$$

$\hookrightarrow$ geometric series

Corollary: (Cauchy's Criteria) The series $\sum a_n$ converges if and only if given $\varepsilon > 0, \exists n_0$ such that

$$\underline{|a_{n+1} + a_{n+2} + \dots + a_{n+p}|} < \varepsilon$$

$\forall n > n_0$.

Proof: $\sum a_n < \infty \iff S_n < \infty \iff S_n$ is Cauchy

A series $\sum a_n$ is absolutely convergent if $\sum |a_n|$ is convergent.

$\boxed{S_n}$

$\sum \boxed{a_n} \boxed{b_n}$

Ex: $\sum \frac{(-1)^n}{n}$; $\sum \frac{1}{n}$ diverges.

converges

$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} \dots$ $\boxed{\frac{1}{20}}$

$s_1 = -1$ $\boxed{\frac{1}{6}}$ $s_2 = \frac{1}{2}$ $S_5 = S_3 + \frac{1}{4} - \frac{1}{5}$

$s_3 = -1 + \frac{1}{2} - \frac{1}{3}$ $S_4 = \frac{1}{2} - \frac{1}{3} + \frac{1}{4}$

$s_1 < s_3 < s_5 < s_6 < \dots \to$ converges

$\frac{1}{2} = s_2 > s_4 > s_6 > \dots > 0 \to$ converges

$\lim s_{2n-1} = a$; $\lim s_{2n} = b$

$s_{2n} - s_{2n-1} = \frac{1}{2n} \to 0$

$\boxed{a = b}$

A series $\sum a_n$ is conditionally convergent if $\sum a_n$ is convergent but $\sum |a_n|$ is divergent.

Theorem: Every absolutely convergent series is convergent

$\sum |a_n| < \infty \overset{?}{\Longrightarrow} \sum a_n$ converges

$n > n_0 \Rightarrow \underline{|a_{n+1}| + |a_{n+2}| + \dots + |a_{n+p}| < \varepsilon}$

$\Rightarrow |a_{n+1} + a_{n+2} + \dots + a_{n+p}|$

$x_n = \{-1, 1, -1, 1, -1, \dots\}$

$\leq |a_{n+1}| + |a_{n+2}| + \dots + |a_{n+p}| < \varepsilon$

$\sum b_n$ converges

Corollary: $n > n_0 \Rightarrow |a_n| \leq c b_n \Rightarrow \sum |a_n| < \infty$

$\underline{\limsup \sqrt[n]{|a_n|} < 1}$

Corollary: (Root test) If $\exists n_0 \in \mathbb{N}$; $\underline{n > n_0 \Rightarrow \sqrt[n]{|a_n|} \leq c < 1}$ then $\sum a_n$ is abosolutely convergent.

$\Downarrow$

$\limsup \sqrt[n]{|a_n|} > 1 \Rightarrow \underline{\sum a_n \text{ diverges}}$

$\sum |a_n| \leq \sum c^n$

Corollary: $\lim \sqrt[n]{|a_n|} < 1 \Rightarrow \sum a_n$ abs. conv.

$\lim \sqrt[n]{|a_n|} > 1 \Rightarrow \sum a_n$ diverges.

Theorem: (Ratio test) $\sum a_n, \sum b_n$; $a_n \neq 0$, $\sum b_n$ converges $\overset{b_n > 0}{}$

If $\exists n_0 \in \mathbb{N}$: $\underbrace{n > n_0 \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| \leq \left|\frac{b_{n+1}}{b_n}\right|}$, then $\sum a_n$ abs. conv.

$$\left|\frac{a_{n_0+2}}{a_{n_0+1}}\right| \leq \left|\frac{\cancel{b_{n_0+2}}}{b_{n_0+1}}\right|$$

$$\left|\frac{\cancel{a_{n_0+3}}}{\cancel{a_{n_0+2}}}\right| \leq \left|\frac{\cancel{b_{n_0+3}}}{\cancel{b_{n_0+2}}}\right|$$

$$\vdots$$

$$\left|\frac{a_n}{\cancel{a_{n-1}}}\right| \leq \left|\frac{b_n}{\cancel{b_{n-1}}}\right|$$

Multiplying them together:

$$\left|\frac{a_n}{a_{n_0+1}}\right| \leq \left|\frac{b_n}{b_{n_0+1}}\right|$$

$$\boxed{|a_n| \leq K \cdot b_n}$$

Corollary: $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$ then $\sum a_n$ abs. conv.

> 1 " " diverges.

Corollary: $\lim \left|\frac{a_{n+1}}{a_n}\right| < 1$ then $\sum a_n$ abs. conv.

> 1 " " diverges

Ex: $e^x = \sum \frac{x^n}{n!} = a_n$ is absol. conv.

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \frac{\cancel{|x|^{n+1}}}{\cancel{(n+1)!}} \cdot \frac{\cancel{n!}}{\cancel{|x|^n}} = \frac{|x|}{(n+1)}$$

$$\lim \frac{|x|}{(n+1)} = 0 < \boxed{1}$$

Theorem: For any bouded sequence a_n positive:

$$\liminf \frac{a_{n+1}}{a_n} \le \underbrace{\liminf \sqrt[n]{a_n} \le \limsup \sqrt[n]{a_n}}_{\lim \sqrt[n]{a_n}} \le \limsup \frac{a_{n+1}}{a_n}$$

"Root test is stronger than the ratio test"

Theorem: (Dirichlet) Let b_n be a nonincreasing sequence of positive numbers with $\lim b_n = 0$ and $\sum a_n$ a series with bounded partial sum "s_n". Then

$$\sum a_n \cdot b_n \text{ converges.}$$

Ex: $\boxed{\sum \frac{(-1)}{n}}$; $\underbrace{a_n = (-1)^n}$ $b_n = \boxed{\frac{1}{n}}$

Corollary: (Abel) $\sum a_n$ convergent; b_n is nonincreasing then $\sum a_n b_n$ converges.

Corollary: (Leibniz) b_n is nonincreasing, $b_n > 0$, $\lim b_n = 0$. Then $\sum (-1) b_n$ converges.

Ex: $\sum \frac{\cos(n \cdot x)}{n} \overset{?}{=} f(x)$ Fourier Analysis

$b_n = \frac{1}{n}$ $\underbrace{a_n = \cos(nx)}$

$\underbrace{e^{ix} = \cos x + i\sin x}$

$\boxed{S_n} \overset{!}{=} \cos x + \cos 2x + \dots + \cos(nx)$

$1 + S_n = \mathrm{Re}\left[\underbrace{1 + e^{ix} + e^{2ix} + \dots + e^{nix}}\right]$

$= \mathrm{Re}\left[\frac{1 - e^{(n+1)ix}}{1 - e^{ix}}\right]$

$|1 + \boxed{S_n}| \leq \boxed{\frac{2}{|1 - e^{ix}|}} \rightarrow$ doesn't depend on "n"

$\sum a_n = a_1 + a_2 + a_3 + a_4 + \dots$

$\sum n = 1 + 2 + 3 + 4 + \dots$

$100 + 5 + \dots$

$a_{100} + a_5 + a_{122} + \dots + \dots$

Theorem: If a series $\sum a_n$ is conditionally convergent then $\forall c \in \mathbb{R}$, it's possible to reorganize $\sum a_n$ in such a way that $\sum a_n = c$.