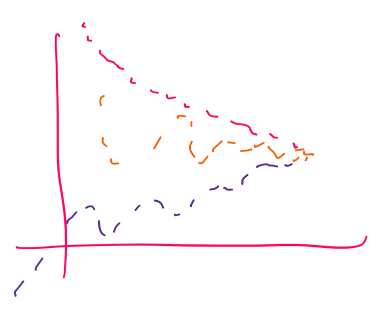
Wednesday, October 9, 2024 3:32 PM

Cor: $x_n \le y_n$ convergent $\Rightarrow \lim x_n \le \lim y_n$

Cor $a \le y_n \Rightarrow \lim y_n \ge a$; $a \in \mathbb{R}$

Theorem: If $x_n \le y_n \le z_n$ and $\lim x_n = \lim z_n$ then

$$\lim y_n = \lim x_n = \lim z_n$$



$\lim x_n = a$
$\Updownarrow$
$x_n \to a$

$a \in \mathbb{R}$ is an accumulation point of x_n if there is a subsequence x_{n_k} such that $x_{n_k} \to a$ ($\lim_{k\to\infty} x_{n_k} = a$)

Ex: $\{0,1,0,1,0,1\}$

$x_{2n-1} \to 0$

$x_{2n} \to 1$

Theorem: $a \in \mathbb{R}$ is an accum. point $\Leftrightarrow$ $\forall \varepsilon > 0$ there are infinitely many "n" such that $x_n \in (a-\varepsilon, a+\varepsilon)$

"Every open interval containing "a" contains infinitely many "x_n"s

Ex: $x_n = a$; i.e $\{a,a,a,a,\dots\}$ / $\lim x_n = a$ — $\liminf x_n = a$, $\limsup x_n = a$

$\to$ $x_n = (-1)^n$, $\{-1,1,-1,1,\dots\}$ — $\liminf x_n = -1$, $\limsup x_n = 1$

$x_n = n$; $\{1,2,3,4,5,6,\dots\}$

$1 < 2 < 3 < 4 < \dots$

Ex: Any $r \in \mathbb{R}$ is an accum. point of a rational sequence.

x_n bounded

$$m \le x_n \le M$$

$$X_n = \{x_n, x_{n+1}, x_{n+2}, x_{n+3}, \dots\}$$

$X_{n+1} \subseteq X_n$

$X_n \subseteq [m, M]$

$a_n = \inf X_n$, $b_n = \sup X_n$

$\liminf x_n = \lim a_n = \sup a_n = a$

$\limsup x_n = \lim b_n = \inf b_n = b$

$a_n \le b_n$

$a - \varepsilon \le a_n$

$a - \frac{1}{k} \le a_k$

$a - \frac{1}{k} \le a_k \le a + \frac{1}{k}$

$a_k \in (a-\varepsilon, a+\varepsilon)$

$$m \le a_1 \le a_2 \le \dots \le a_n \le \dots \le b_n \le b_{n-1} \le \dots \le b_2 \le b_1 \le M$$

Theorem: Let x_n be a sequence. Then $\liminf x_n$ is the smallest accumulation point, and $\limsup x_n$ is the largest one.

Proof: We prove the claim about $\liminf x_n$.

Corollary: (Bolzano-Weierstrass theorem):
Every bounded sequence has a convergent subsequence.

Corollary: x_n is convergent if and only if $\liminf x_n = \limsup x_n$.

Cauchy-sequences

Functional Analysis

x_n is Cauchy if $\forall \varepsilon > 0, \exists n_0$:

$$m, n > n_0 \Rightarrow |x_n - x_m| < \varepsilon$$

Theorem: If x_n is convergent then x_n is Cauchy.

Sketch: $a \in \mathbb{R}$; $\lim x_n = a$ — $n > n_1 \Rightarrow |x_m - a| < \frac{\varepsilon}{2}$

$\forall \varepsilon > 0 ; \exists n_0 \in \mathbb{N} : n > n_0 \Rightarrow |x_n - a| < \frac{\varepsilon}{2}$

$$|x_n - x_m| = |x_n - a + a - x_m|$$
$$\le \underbrace{|x_n - a|}_{\frac{\varepsilon}{2}} + \underbrace{|a - x_m|}_{\frac{\varepsilon}{2}} = \varepsilon$$

$\max\{n_0, n_1\}$

Theorem: If x_n is Cauchy then it is convergent.

lemma: Every cauchy sequence is bounded.

lemma: If a cauchy sequence x_n has a convergent subsequence x_{n_k}, $\lim x_{n_k} \to a$ then x_n is convergent and $x_n \to a$.

Corollary: x_n is Cauchy $\Leftrightarrow$ x_n is convergent.

Infinite limits

$\lim x_n = +\infty$ (converges to infinity)

$\forall M > 0 ; \exists n_0 \in \mathbb{N} : n > n_0 \Rightarrow x_n > M$

$\lim x_n = -\infty$ (converges to neg. inf.)

$\forall M > 0 ; \exists n_0 \in \mathbb{N} : n > n_0 \Rightarrow x_n < -M$

Ex: $\{1,2,3,4,5,\dots\}$, $x_n = n$

$\lim n = +\infty$

$x_n = (-1)^n$; $\{-1, 1, -1, 1, \dots\}$

$x_n = (-1)^n \cdot n$ $\{-1, 2, -3, 4, -5, \dots\}$

Theorem: $\lim x_n = +\infty$ an y_n is bounded

$\lim(x_n + y_n) = \lim x_n + \lim y_n$

$\lim(x_n \cdot y_n) = \lim x_n \cdot \lim y_n$

$$x_n > 0 \quad \lim x_n = 0 \Leftrightarrow \lim \frac{1}{x_n} = +\infty$$

Ex: $x_n = \sqrt{n+1}$ — $\lim(x_n + y_n) \ne \lim x_n + \lim y_n$

$y_n = -\sqrt{n}$ — $\cancel{\lim x_n + y_n = \infty - \infty = "0"}$

$\lim x_n = +\infty$

$\lim y_n = -\infty$

$$\lim \sqrt{n+1} - \sqrt{n} = \lim \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})}$$
$$= \lim \frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}}$$
$$= \lim \frac{1}{\sqrt{n+1}+\sqrt{n}}$$
$$= 0$$

indeterminate

$\infty - \infty$

$x_n = n^2$ — $\lim n^2 = +\infty$

$y_n = -n$ — $\lim -n = -\infty$

$\lim n^2 - n = \lim n \cdot (n-1) = \lim n \cdot \lim (n-1)$
$= +\infty$

$\frac{\infty}{\infty}$

Ex: $x_n = [2 + (-1)^n] n$

$y_n = n$ — $\lim \frac{x_n}{y_n} = \lim [2 + (-1)^n]$ doesn't exist

$\lim y_n = +\infty$

$\lim x_n = +\infty$

Ex: $a > 1 \Rightarrow a = 1 + x$ — $(1+x)^n \ge 1 + n \cdot x$

$(1+x)^n = 1 + n \cdot x + \frac{n \cdot (n-1)}{2} x^2$

$$\lim \frac{a^n}{n} = +\infty$$

$a^n = (1+x)^n \ge 1 + n \cdot x + \frac{n \cdot (n-1)}{2} x^2$

$+\infty \ge \lim \frac{a^n}{n} \ge \lim\left(\cancel{\frac{1}{n}} + x + \frac{(n-1)}{2} x^2\right) = +\infty$
$\; = +\infty$

Series

a priori

Given a sequence "x_n", we want to give meaning to expressions of the form:

$$x_1 + x_2 + x_3 + x_4 + \dots$$

$\sum \frac{1}{n}$

$S_n = x_1 + x_2 + \dots + x_n$ (Partial sums)

$$\boxed{\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} S_n}$$

x_n $\to$ general term

exists ; $\lim s_n = a$; converges

doesn't exist ; diverges

Theorem: If $\sum a_n$ converges then $\lim a_n = 0$.

$$a_n = S_n - S_{n-1}$$

$$\lim a_n = \lim s_n - s_{n-1} = \lim s_n - \lim s_{n-1} = 0$$

Ex: $\sum \frac{1}{n}$; $\lim \frac{1}{n} = 0$; It's enough to find a divergent subsequence of S_n :

$\lim S_n = \sum \frac{1}{n}$

$$S_{2^n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^n}$$
$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots$$
$$> 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \dots + \frac{2^{n-1}}{2^n}$$

$$S_{2^n} > \underbrace{1 + n \cdot \frac{1}{2}}_{\text{diverges}}$$

$\therefore \lim S_n = \sum \frac{1}{n}$ does not converge.

Ex: (Geometric Series)

$$\sum a^n$$

$|a| \ge 1$ diverges $a = 1$; $S_n = n$

$|a| < 1$

$$S_n = \frac{1 - a^{n+1}}{1 - a}$$

$\lim S_n = \frac{1}{1-a}$ $\therefore \sum a^n = \frac{1}{1-a}$; $|a| < 1$

Properties

$\sum a_n, \sum b_n$ converge $\Rightarrow$ $\sum (a_n + b_n)$ converge $= \sum a_n + \sum b_n$

$\sum a_n < +\infty \Rightarrow \sum c a_n < +\infty$ $= c \cdot \sum a_n$

$\sum a_n, \sum b_n$ converge

$$c_n = \sum_{i=1}^{n} a_i b_n + \sum_{j=1}^{n-1} a_n b_j$$

$$\sum c_n = \sum a_n \cdot \sum b_n$$