Final Exam

1. Let $f:\mathbb{R}\to\mathbb{R}$ be continuous and $X\subseteq\mathbb{R}$. Show that if $f(x)=0,\ \forall x\in X$ then $f(x)=0,\ \forall x\in\overline{X}$. Here, $\overline{X}$ denotes the closure of X.

   Solution. Let $x\in\overline{X}$, then by definition, we can find a sequence $x_n\in X$ such that $x_n\to x$, since f is continuous, $f(x_n)\to f(x)$. Since $f(x_n)=0$, the result follows. □

2. Let $f:[0,1]\to\mathbb{R}$ be continuous such that $f(0)=f(1)$. Show that there exists $c\in[0,\frac{1}{2}]$ such that $f(c)=f(c+\frac{1}{2})$.

   Solution. Set $g(x)=f(x)-f(x+\frac{1}{2})$, then $g(0)=f(0)-f(\frac{1}{2})=-g(\frac{1}{2})$. If $g(0)=-g(\frac{1}{2})=0$ then the statement is obvious, suppose $g(0)\neq 0$, then by the Intermediate value theorem there exists $c\in[0,\frac{1}{2}]$ such that $g(c)=0$. □

3. A function $f:\mathbb{R}\to\mathbb{R}$ is periodic if there exists $p\in\mathbb{R}$ such that $f(x+p)=f(x)$ for every $x\in\mathbb{R}$. Show that every continuous periodic function is bounded. *[Hint: Use the Extreme Value Theorem.]*

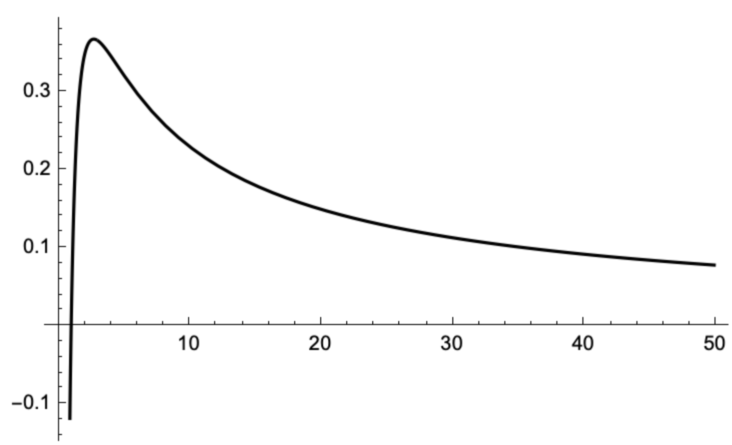
   Solution. It suffices to prove that f restricted to $[0,p]$ is bounded since $f(x+p)=f(x)$. Since $[0,p]$ is compact, the result follows from Weirstrass Extreme Value Theorem. □

4. Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable such that for $c,x\in\mathbb{R}$ we have $f(cx)=cf(x)$. Show that $f(x)$ satisfies
$$f(x)=f'(0)x.$$

   Solution. Take $x=1$, then $f(c)=cf(1)$ for every $c\in\mathbb{R}$, or $f(x)=xf(1)$. Taking the derivative on both sides gives $f'(x)=f(1)$, hence $f(x)=f'(0)x$. □

5. Let $f:\mathbb{R}^+\to\mathbb{R}$ be given by $f(x)=\frac{\ln x}{x}$. Give the intervals where f increases, and when it decreases. Also, find all the critical points of f. Recall that a point $a\in\mathbb{R}$ is called a critical point of the function $f(x)$, if $f'(a)=0$.

   Solution. A quick computation shows that $f'(x)=\frac{1-\ln x}{x^2}$, hence $x=e$ is the only critical point (local max) and f is increasing on $(0,e)$ and decreasing on $(e,+\infty)$.

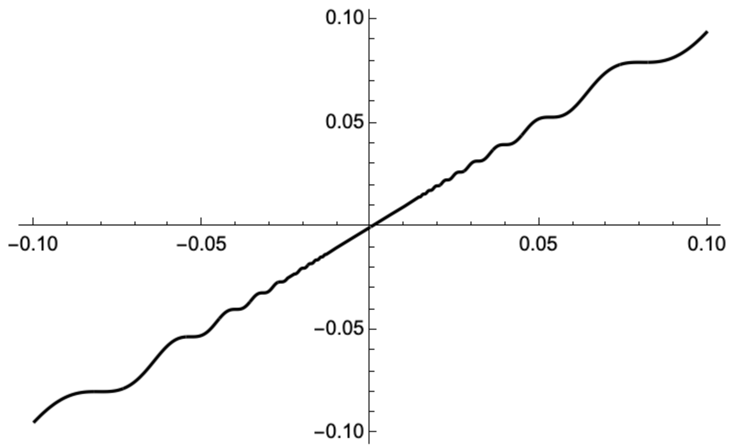


□

Extra (2 pts). Show that $x = 0$ in the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 \sin\frac{1}{x} + x$, if $x \neq 0$, and $f(0) = 0$, is a counter-example to the statement:

'*The limit of a sequence of critical points is itself a critical point.*'

Solution. A quick computation shows that $f'(0) = 1$ and $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x} + 1$ if $x \neq 0$. Therefore, $x = 0$ is not a critical point but $x_n = \frac{1}{2\pi n}$ is a sequence of critical points converging to 0.



□