Exercises

8. Find the set of adherent points at 0 of the function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ be given by $f(x) = \frac{\sin\left(\frac{1}{x}\right)}{1+e^{\frac{1}{x}}}$.

   Solution. We claim the set of adherent points at 0 is $[-1,1]$. Notice that

   $$\left|\frac{\sin\left(\frac{1}{x}\right)}{1+e^{\frac{1}{x}}}\right| \leq 1.$$

   It suffices to show that any $c \in [-1,1]$ is adherent. Let $a \in \mathbb{R}$ be a number satisfying $\sin a = c$. Then the sequence $x_n := -\frac{1}{a+2n\pi}$ satisfies $x_n \to 0$ and $f(x_n) \to c$. □

10. Given a nonempty compact set $K \subseteq \mathbb{R}$ and a point $a \in \mathbb{R}$. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ whose the set of adherent points at a is K.

    Solution. Let $\{y_1, y_2, \ldots\} \subseteq K$ be a countable dense subset of K and $x_n = a + \frac{1}{n}$. Then

    $$f(x) = \begin{cases} y_n, & x = x_n \\ y_1, & x \neq x_n \end{cases}$$

    is an example of function having K as adherent points. □

11. Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by

    $$f(x) = \begin{cases} x, & x \notin \mathbb{Q} \\ 0, & x = 0 \\ q, & x = \frac{p}{q} \text{ and } gcd(p,q) = 1, p > 0 \end{cases}$$

    Show that f is unbounded on any non-degenerate interval.

    Solution. Let (a,b) be a non-degenerate interval. Consider the sequence

    $$x_n = a + \frac{b-a}{n} = \frac{an + (b-a)}{n}$$

    For each n,if $gcd(an + (b-a), n) > 1$ drop the term x_n. The remaning elements for a subsequence x_{n_k} satisfying $f(x_{n_k}) \to +\infty$. □

15. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that the *zero set* of f

    $$Z(f) = \{x; f(x) = 0\}$$

    is a closed set. Conclude that if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous then the zero set $Z(f-g)$ is closed.

Solution. Let $x_n \in Z(f)$ be convergent sequence $x_n \to a$. Since f is continuous, $f(x_n) \to f(a)$. But $f(x_n) \equiv 0$, thus $f(a) = 0$ and $a \in Z(f)$. It follows that $\overline{Z(f)} = Z(f)$. The conclusion follows from the fact that $h := f - g$ is continuous, hence $Z(h)$ is closed. □

16. Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x + ax\sin(x)$. Show that

$$|a| < 1 \Rightarrow \lim_{x \to \pm\infty} f(x) = \pm\infty.$$

Solution. Notice that $(1-a)x \le f(x) \le (1+a)x$. Let $M > 0$ be given, if $x > \frac{M}{(1-a)}$ then $f(x) > M$. Hence, $\lim_{x\to\infty} f(x) = +\infty$. Similarly, if $x < -\frac{M}{(1+a)}$ then $f(x) < -M$. □

21. Let $S \subseteq \mathbb{R}$ be nonempty. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \inf\{|x - s|; s \in S\}$$

Show that f is *Lipschitz*: $\forall x, y \in \mathbb{R} \Rightarrow |f(x) - f(y)| \le |x - y|$.

Solution. Let $x, y \in S$. For any $s \in S$:

$$|x - s| \le |x - y| + |y - s|.$$

Taking the infimum over S we get

$$f(x) \le |x - y| + f(y)$$

Swapping x and y we obtain $f(y) \le |x - y| + f(x)$ and the result follows. □

24. Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function and for each $n \in \mathbb{N}$ consider the set C_n of all points $a \in \mathbb{R}$ satisfying: there exists an open interval $I \ni a$ containing a, such that

$$x, y \in I \Rightarrow |f(x) - f(y)| < \frac{1}{n}.$$

(a) Show that each C_n is open.

Solution. Let $a \in C_n$ then by definition there exists an open interval I such that $\forall b \in I : x, y \in I \Rightarrow |f(x) - f(y)| < \frac{1}{n}$, hence $I \subseteq C_n$ and C_n is open. □

(b) Show that f is continuous at a if and only if $a \in C_n$ for every $n \in \mathbb{N}$. Conclude that the set of all points where f is continuous can be written as a countable intersection of open sets. In particular, there is no function $f : \mathbb{R} \to \mathbb{R}$ who is discontinuous only at $\mathbb{R} - \mathbb{Q}$. (See exercise 35 from the previous chapter)

Solution. If f is continuous at a then $\lim_{x\to a} f(x) = f(a)$ implies $a \in C_n$. Conversely, if $a \in \cap C_n$, then given $\epsilon > 0$, choose n such that $\frac{1}{n} < \epsilon$. Then $x \in I \Rightarrow |f(x) - f(a)| < \epsilon$, and f is continuous at a. □

25. Show that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ that takes every rational number to an irrational number, and vice-versa.

Solution. Suppose there is such a function, say $f(\mathbb{R} - \mathbb{Q}) \subseteq \mathbb{Q}$. Since f is continuous $f(\mathbb{R})$ is an interval. On the other hand, $f(\mathbb{R}) = f(\mathbb{R} - \mathbb{Q}) \cup f(\mathbb{Q}) \subseteq \mathbb{Q} \cup f(\mathbb{Q})$ is countable, a contradiction. □

27. Show that $f : (-1, 1) \to \mathbb{R}$ given by $f(x) = \frac{x}{1-|x|}$ is a homeomorphism. The graph of f is depicted below.

Solution. Since f is the quotient of continuous functions it is itself continuous. We can easily check that the inverse $f : \mathbb{R} \to \mathbb{R}$ is given by the continuous function

$$f^{-1}(x) = \frac{x}{1 + |x|},$$

hence f is a homeomorphism. □

34. Show that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin x$ is uniformly continuous, but $g(x) = \sin x^2$ is not.

Solution. Notice that if $x_n - y_n \to 0$ then

$$\sin x_n - \sin y_n = 2 \cos \frac{x_n + y_n}{2} \sin \frac{x_n - y_n}{2} \to 0.$$

It follows that $\sin x$ is uniformly continuous. Now, consider the sequences $x_n = \sqrt{2\pi n}$ and $y_n = \sqrt{\frac{pi}{2} + 2\pi n}$. We have $x_n - y_n \to 0$ but $f(x_n) - f(y_n) = 0 + 1 = 1$. □