Final Exam - Fall 2025

1. Show that if there are $r > 0$ and $k, n_0 \in \mathbb{N}$ such that

$$n > n_0 \Rightarrow r \leq x_n \leq n^k$$

for some sequence x_n, then $\lim \sqrt[n]{x_n} = 1$. Conclude that $\lim \sqrt[n]{\ln n} = 1$.

Solution. We have $\sqrt[n]{r} \leq \sqrt[n]{x_n} \leq (\sqrt[n]{n})^k$. The result follows by the Squeeze Theorem. For the second statement, take $x_n = \ln n$ and notice that $\ln 2 \leq \ln n \leq n$ for $n \geq 2$.

2. Determine if the series

$$\sum_{n=1}^{+\infty} \left(\frac{\ln n}{n}\right)^n$$

converges.

Solution. Notice that

$$\sqrt[n]{\left(\frac{\ln n}{n}\right)^n} = \frac{\ln n}{n},$$

and $\frac{\ln n}{n} \to 0$ as $n \to +\infty$. By the root test, the series converges.

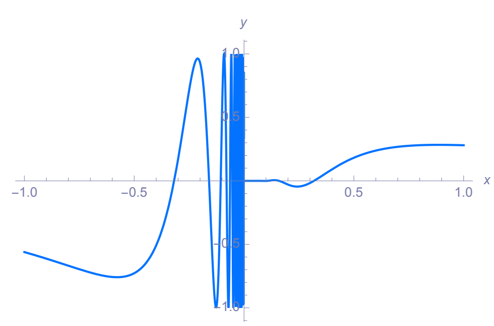
3. Show that every uncountable set $X \subseteq \mathbb{R}$ has an accumulation point.

Solution. Suppose not, then all the points of X are isolated. Let D be a countable dense subset of X. Take $x \in X$, since x is isolated there exists a neighborhood of x containing only x itself, but since D is dense, we must have $x \in D$. Hence, $X = D$ is countable, a contradiction.

4. Consider the function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ defined by

$$f(x) = \frac{\sin \frac{1}{x}}{1 + 2^{\frac{1}{x}}}.$$

Determine the adherent values of f at 0 and conclude that $\liminf_{x \to 0} f = -1, \limsup_{x \to 0} f = 1$. The graph of f is depicted below.



Solution. By looking at the graph we can clearly see that the set of adherent values is $[-1,1]$. Indeed, let $c \in [-1,1]$ and define $x_n = -\frac{1}{\sin^{-1}(-c)+2\pi n}$. Then $x_n \to 0$ and $f(x_n) \to c$. The second statement follows from the fact that $\liminf$ and $\limsup$ are the smallest and largest of the adherent values respectively.

5. Let $X \subseteq \mathbb{R}$ be a set with the following property: Every function $f : X \to \mathbb{R}$ with domain X is uniformly continuous. Show that X is closed (but not necessarily compact, since every function defined over $\mathbb{N}$ is uniformly continuous and yet $\mathbb{N}$ is not compact).

   Solution. Suppose not, let $a \in \mathbb{R}$ be adherent to X but not in X. Consider the function $f : X \to \mathbb{R}$ given by $f(x) = \frac{1}{x-a}$. We claim f is not uniformly continuous at a. Indeed, $a \in X'$ but $\lim\limits_{x\to a} f(x)$ doesn't exist, hence f cannot be uniformly continuous at a, a contradiction.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{x^2}{1+x^4}$. Compute $f^{(2025)}(0)$.

   Solution. Recall that
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$
   so
$$\frac{1}{1+x^4} = \sum_{n=0}^{\infty} (-1)^n x^{4n},$$
   thus
$$\frac{x^2}{1+x^4} = \sum_{n=0}^{\infty} (-1)^n x^{4n+2}.$$
   Hence $f(x)$ agrees with its Taylor Series at $x = 0$. Notice that $2025 = 4 \times 506 + 1$, in particular, 2025 is not of the form $4n + 2$, thus $f^{(2025)}(0) = 0$.

7. Show that the function $f : [a,b] \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \cap [a,b] \\ x+1, & \text{if } x \in (\mathbb{R} - \mathbb{Q}) \cap [a,b] \end{cases}$$
   is not integrable and compute its lower and upper integral.

   Solution. Let P be a partition of $[a,b]$. Notice that $s(f,P) = s(x,P)$, hence $\underline{\int_a^b} f = \int_a^b x = \frac{b^2-a^2}{2}$. Similarly, $S(f,P) = S(x+1,P)$, and $\overline{\int_a^b} f = \int_a^b x + 1 = \frac{b^2-a^2}{2} + (b-a)$.

8. Show that the integral
$$\int_0^{+\infty} x \sin x^4 \, dx$$
   is convergent despite the fact that $f(x) = x \sin x^4$ is unbounded on $[0,+\infty]$.

Solution. Since $x \sin x^4$ is continuous, the integral $\int_0^1 x \sin x^4 \, dx$ is finite. It suffices to show that $\int_1^{+\infty} x \sin x^4 \, dx$ converges.

Notice that

$$\int_1^{+\infty} x \sin x^4 \, dx = \int_1^{+\infty} \frac{\sqrt[4]{u}}{4u^{\frac{3}{4}}} \sin u \, du = \int_1^{+\infty} \frac{1}{4u^{\frac{1}{2}}} \sin u \, du$$

Integrating by parts the last integral, we obtain

$$\int_1^{+\infty} \frac{1}{4u^{\frac{1}{2}}} \sin u \, du = \frac{1}{4u^{\frac{1}{2}}}(-\cos u)|_1^{+\infty} - \int_1^{+\infty} \frac{1}{8u^{\frac{3}{2}}} \cos u \, du = \frac{\cos 1}{4} - \frac{1}{8} \int_1^{+\infty} \frac{1}{u^{\frac{3}{2}}} \cos u \, du$$

and the latter integral converges by comparison with $\int_1^{+\infty} \frac{1}{u^{\frac{3}{2}}} \, du$.