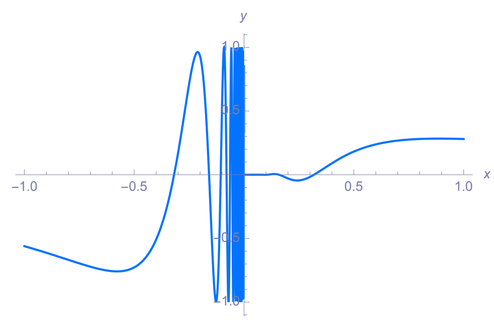
Final Exam - Fall 2025

1. Show that if there are $r > 0$ and $k, n_0 \in \mathbb{N}$ such that
$$n > n_0 \Rightarrow r \leq x_n \leq n^k$$
for some sequence x_n, then $\lim \sqrt[n]{x_n} = 1$. Conclude that $\lim \sqrt[n]{\ln n} = 1$.

2. Determine if the series
$$\sum_{n=1}^{+\infty} \left(\frac{\ln n}{n}\right)^n$$
converges.

3. Show that every uncountable set $X \subseteq \mathbb{R}$ has an accumulation point.

4. Consider the function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ defined by
$$f(x) = \frac{\sin \frac{1}{x}}{1 + 2^{\frac{1}{x}}}.$$
Determine the adherent values of f at 0 and conclude that $\liminf\limits_{x\to 0} f = -1, \limsup\limits_{x\to 0} f = 1$. The graph of f is depicted below.



5. Let $X \subseteq \mathbb{R}$ be a set with the following property: Every function $f : X \to \mathbb{R}$ with domain X is uniformly continuous. Show that X is closed (but not necessarily compact, since every function defined over $\mathbb{N}$ is uniformly continuous and yet $\mathbb{N}$ is not compact).

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{x^2}{1+x^4}$. Compute $f^{(2025)}(0)$.

7. Show that the function $f : [a, b] \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \cap [a, b] \\ x + 1, & \text{if } x \in (\mathbb{R} - \mathbb{Q}) \cap [a, b] \end{cases}$$
is not integrable and compute its lower and upper integral.

8. Show that the integral
$$\int_0^{+\infty} x \sin x^4 \, dx$$
is convergent despite the fact that $f(x) = x \sin x^4$ is unbounded on $[0, +\infty]$.