Exam III

Choose 4 questions below.

Topology

1. Prove that for every $X \subseteq \mathbb{R}$ we have
$$\mathbb{R} - X^\circ = \overline{\mathbb{R} - X},$$
where X° is the interior of X.

   Solution. Notice that $a \in \mathbb{R} - X^\circ \iff \forall \delta > 0, (a-\delta, a+\delta) \cap (\mathbb{R} - X) \neq \emptyset \iff a \in \overline{\mathbb{R} - X}$.

2. Let $K \subseteq \mathbb{R}$ be compact. Show that if all the points of K are isolated then K is finite. *Hint: Remember that every cover of K has a finite subcover.*

   Solution. For each $x \in K$ there exists an open interval I_x such that $I_x \cap K = \{x\}$. On the other hand the collection of I_x form an open cover of K, $K = \bigcup_{x \in K} I_x$. Since K is compact, we may choose a finite subset $F \subseteq K$ such that $K = \bigcup_{x \in F} I_x$. But $I_x = \{x\}$, this implies $K = F$.

3. Show that $\frac{1}{10}$ is an element of the Cantor set. *Hint: Consider the base 3 expansion of $\frac{1}{10}$.*

   Solution. $\frac{1}{10} =_3 .\overline{022}$. Since its expansion in base 3 does not contain 1, $\frac{1}{10}$ is in the Cantor set.

4. Prove that the set of endpoints of the intervals removed in the construction of the Cantor set K is dense (and countable) in K.

   Solution. Let $X = \{x$;x is a removed endpoint$\}$. Given any open interval (a, b) containing $c \in K$, after the n-th iteration, only intervals of length $\frac{1}{3^n}$ will remain, in particular, when $\frac{1}{3^n} < b - a$, the interval (a, b) will contain a removed endpoint.

Limits and Continuity

5. Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for all $X \subseteq \mathbb{R}$, $f(\overline{X}) \subseteq \overline{f(X)}$. *Hint: For the converse, it's easier to show the contrapositive. Namely, suppose f is discontinuous at a and show that $f(\overline{X}) \not\subseteq \overline{f(X)}$.*

   Solution. Let $y \in f(\overline{X})$. Then $y = f(x)$ for some $x \in \overline{X}$. hence, there exists a sequence $x_n \in X$ such that $x_n \to x$. Since f is continuous, $f(x_n) \to f(x) = y$, and it follows that $y \in \overline{f(X)}$.

   Conversely, suppose f is discontinuous at a then there exist $\epsilon > 0$ and a sequence x_n such that $x_n \to a$ but $|f(x_n) - f(a)| \geq \epsilon$. Set $X = \{x_n\,;\, n \in \mathbb{N}\}$. Then $a \in \overline{X}$ but $f(a) \notin \overline{f(X)}$, thus $f(\overline{X}) \not\subseteq \overline{f(X)}$.

6. Let $f : [0,1] \to \mathbb{R}$ be a continuous function satisfying $f(0) = f(1)$. Show that there exists $c \in [0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$. *Hint: Consider the function* $g(x) = f(x) - f(x + \frac{1}{2})$ *and use the Intermediate Value Theorem.*

   Solution. The function function $g : [0, \frac{1}{2}] \to \mathbb{R}$ defined by $g(x) = f(x) - f(x + \frac{1}{2})$, satisfies $g(0) = f(0) - f(\frac{1}{2}) = -(f(\frac{1}{2}) - f(0)) = -g(1)$. By the Intermediate Value Theorem, there exists $c \in [0, \frac{1}{2}]$ such that $g(c) = 0$.

7. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is periodic if there exists a real number $p > 0$ such that for every $x \in \mathbb{R}$

$$f(x + p) = f(x).$$

   Show that every continuous periodic function is bounded and achieves its maximum and minimum values, i.e. $\exists a, b \in \mathbb{R},\ \forall x \in \mathbb{R} : f(a) \leq f(x) \leq f(b)$. *Hint: Extreme Value Theorem.*

   Solution. Notice that the function $g : [0, p] \to \mathbb{R}$ defined by $g(x) = f(x)$ is continuous and defined on the compact $[0, p]$. By the Extreme Value Theorem, there exist $a, b \in [0, p]$ such that $g(a) \leq g(x) \leq g(b)$. Now, given any $y \in \mathbb{R}$, since f is periodic, one can find $x \in [0, p]$, such that $f(y) = f(x) = g(x)$. Hence, $f(a) \leq f(y) \leq f(b)$ for every $y \in \mathbb{R}$.

8. Show that the function $f : (0, 1] \to (1, +\infty)$ given by $f(x) = \frac{1}{x}$ is not uniformly continuous. *Hint: Use the definition or analyze* $\lim_{x \to 0} f(x)$.

   Solution. Recall that if $f : X \to \mathbb{R}$ is uniformly continuous and $a \in X'$ then $\lim_{x \to a} f(x)$ exists. But since 0 is an accumulation point of $(0, 1]$ and $\lim_{x \to 0} \frac{1}{x}$ does not exists, f can't be uniformly continuous.

Extra

Show there is no continuous function $f : [0,1] \to \mathbb{R}$ with the following property: f achieves each one of its values $f(x)$, $x \in [0, 1]$, exactly twice. *Hint: Argue by contradiction. Use the Extreme Value Theorem.*

Solution. Suppose not, namely, f achieves each one of its values twice. f is obviously not constant. By the Extreme Value Theorem, f achieves its maximum and minimum. Since each extreme value occurs exactly twice, at least one of them is reached at an interior point, say the maximum.

Let's suppose a and b are maximum, with $M = f(a) = f(b)$ and $a \neq 0, 1$. Then there exists $\delta > 0$ such that on the intervals $[a - \delta, a)$, $(a, a + \delta]$ and $[b - \delta, b)$ (if $b = 0$, take $[b, b + \delta)$) we have $f(x) < M$. If we set $m := \max\{f(a - \delta, a), f(a, a + \delta), f(b - \delta, b)\}$, then by the Intermediate value theorem there are points $c \in [a - \delta, a), d \in (a, a + \delta], e \in [b - \delta, b)$ such that $m = f(c) = f(d) = f(e)$, a contradiction